Sergio Bittanti ed.

COUNT RICCATI AND THE EARLY DAYS OF THE RICCATI EQUATION

Pitagora Editrice Bologna

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Alexander Longhi deli *Innocens Alessandri Sculp. Venetiis*

PREFACE

For many years, I studied the Riccati Equation knowing only a little about its origin. The same probably applies to other scientists who analyzed the equation, here and there in the world.

During the organization activity of the workshop on "The Riccati Equation in Control, Systems, and Signals" (Como, Italy, June 26-28, 1989), I came across some historical papers on Count Riccati and his equation(s). This raised my interest, so that I read more and more papers (many of which in Italian or Latin) on the subject.

In the attempt of serving all those who are too much involved in analytic research to spend time on historical issues, I eventually decided to report the results of my reading in this booklet.

Here, the reader will first find the original paper by Jacopo Francesco Riccati published in 1724. In this paper, Riccati reduces a second order differential equation to a first order one, an equation which we would now call a Riccati Equation. The known term of this equation is a power of the independent variable. At the end of the paper, the question was posed of *finding all exponents of the known term for which the separation of variables was possible. From various documents, it is apparent that Riccati already knew infinitely many cases for which the separation of variables was actually possible. His main concern was looking for the most general solution of the problem.*

Riccati's problem was considered by many of the most famous mathematicians. Here, two papers are reprinted; the first one is an Euler's paper of 1764, while the second one is due to Liouville and was published in 1841.

Finally, a paper summarizing my views on the history and prehistory of the Riccati equation can be found.

The support of the C.N.R.-Consiglio Nazionale delle Ricerche is gratefully acknowledged.

Sergio Bittanti
Milano, Via Rivoli
April 1989

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ANIMADVERSIONES IN ÆQUATIONES differentiales secundi gradus, Autore Co. JACOBO RICCATO.

REductio æquationum differentialium secundi ordinis plerumque est adeo perplexa, atque involuta, ut Analystam minus attentum frequentissime eludat. Dum syntheticæ viæ insistimus, & a primis fluxionibus ad altiorem gradum ascendimus, cum assumatur tanquam constans vel nota differentia, vel nulla, eæ difficultates, de quibus sermo erit, vix occurrunt; quæ tamen evitari nequeunt, si problema aliquod proponatur secunda

ſecunda elementa involvens & analytica methodo procedendum ſit. Infinitas dari formulas differentio - differentiales, ad quas pervenitur, nulla adhibita conſtante, nemo profecto ignorat: totidem quoque exhiberi poſſe, ad quas pervenire non conceditur, niſi conſtante in ſubſidium vocata, acutiores non latet Analyſtas: at quomodo ab invicem dignoſci queant, & qua ratione tractandæ ſint, non ita compertum neque obvium puto; cum tamen ſublimioris Geometriæ officium ſit inſpicere, quousque, & quibus in circumſtantiis expreſſiones iſtæ ſolutionem admittant.

Sit ex. gr. conſtruenda curva, in qua quælibet abſciſſæ dignitas ſe habeat directe ut ſecunda differentia ordinatæ, & inverſe ut ſimilis differentia ejusdem abſciſſæ, quæ curva exponitur per æquationem differentialem ſecundi ordinis $x^m ddx = ddy$: ajo nullam curvam inter poſſibiles quæſtioni ſatisfacere, ſi a primis ad ſecundas fluxiones fiat tranſitus, absque eo quod aliqua prima differentia uſurpetur pro conſtante, nec juvabit, ſalva æqualitate, æquationes ipſas quomodocunque alterare ſive per additionem terminorum æqualium, ſive per valorum ſubſtitutionem: at ex oppoſito, conſtante determinata, inveniuntur quidem curvæ problematis conditionem implentes, ſed numero infinitæ, & indole differentes; utpote quæ variantur ad arbitrariæ conſtantis mutationem. Poſteriores haſce expreſſiones, quæ ſub falſa ſpecie nobis imponunt, a veris, atque legitimis ſecernere, videtur eſſe profundioris indagationis; nihilominus certum, & quantum ſubjecta materia patitur, generale Criterium Mathematicis examinandum propono, quod ſaltem uſui erit in his omnibus caſibus, in quibus nos calculus integralis haud deſerit.

Porro ad formulas primis tantum differentiis implicitas revocantur æquationes omnes differentiales ſecundi gradus, ad quas, ſive aſſumta ſive non aſſumta conſtante, perventum eſt, & in quibus ſecundæ fluxiones cum primis, & cum finitis magnitudinibus quomodocunque miſcentur, dummodo alterutra ex indeterminatis fluentibus cum ſuis functionibus æquationem propoſitam non ingrediatur; quod dicendum pariter de illis expreſſionibus, quæ ad hanc formam aliquà adhibita induſtria redigi poſſunt: cæterum in reliquis, quas progreſſus noſter non complectitur, ad aliquos caſus particulares ſe extendere poteſt Analyſtarum ſolertia; at ſi quis canonem generalem inveniret, is profecto eſſet mihi magnus Apollo. Interim conſideranda venit æquatio catholica (A) $zdx=dy$, in qua omnes formulæ differentiales primi gradus continentur; cum litera z deſignet magnitudinem utcunque datam per functiones coordinatarum x & y. Tranſeo ad altiores differentias, nulla aſſumta conſtante, proditque æquatio (B) $zddx+dzdx=ddy$, quæ, dum in eodem ſtatu permanet, nullo negotio integratur. Quod ſi ipſius forma immutetur, ſubrogatis valoribus ab expreſſione (A) mutuo acceptis, tunc infinitæ formulæ oriuntur, quæ majus artificium poſtulant. A ſimplicioribus exemplum peto: loco ipſius dx ſubſtituatur quantitas æqualis $dy:z$, & primus terminus æquationis (B) multiplicetur per dignitatem $z^m dx^m$, reliqui vero per æquivalentem dy^m, unde reſultet nova æquatio (D) $z^{m+1}dx^m ddx+dy^{m+1}dz:z=dy^m ddy$.

Hujusmodi formulæ expedite reducuntur ope alicujus conſtantis, quod ut fiat quantum fieri poteſt, generaliter deſigno pro conſtante fluxionem $dx:q$, eſt autem q magnitudo quomodocunque data per indeterminatas x, y, & conſtantes. Pono $dx:q=dp$, & cum ſit $dx:q$ conſtans, erit pariter conſtans dp. Hinc in æquatione $dx=qdp$ transeundo ad ſecundas differentias habebimus $ddx=dqdp$. Præterea ſtatuo $dy=udp$, & ſumtis ſecundis differentiis in eadem hypotheſi conſtantis dp, erit $ddy=dudp$. Subrogatis in expreſſione (D) valoribus ut ſupra determinatis, & inventis, orietur æquatio $z^{m+1}q^m dqdp^{m+1}+u^{m+1}dz:z\times dp^{m+1}=u^m du\,dp^{m+1}$, & dividendo per dp^{m+1}, $z^{m+1}q^m dq+u^{m+1}dz:z=u^m du$, & ſummando per regulas vulgares, non omiſſa conſtantis g additione, $g+q^{m+1}:\overline{m+1}=u^{m+1}:\overline{\overline{m+1}\times z^{m+1}}$, quæ æquatio dat $u=z\times\overline{q^{m+1}+gm+g}^{\frac{1}{m+1}}$, & quia $dy=udp=udx:q$, opportuna adhibita ſubſtitutione, occurrit æquatio reducta (E) $dy=zdx:q\times\overline{q^{m+1}+gm+g}^{\frac{1}{m+1}}$.

Ex

Ex hoc operandi modo ſponte fluunt nonnulla conſectaria.

Si determinata magnitudine z, æquatio (E) conſtruatur 1.
ſaltem per quadraturas quando fieri poteſt, & indeterminatæ ſunt ſeparabiles, manifeſtum puto curvas infinitas noſtræ formulæ reſpondere, variatur enim natura curvæ, ob mutatam conſtantem $dx:q$, & quilibet valor quantitatis q novam ſemper æquationem localem, ſive algebraicam, ſive tranſcendentem ſubminiſtrat.

Quanquam alterato valore magnitudinis q curvæ diverſæ 2.
originem ducant, certum tamen eſt in quacunque hypotheſi locum invenire inter ipſas curvam, ut ita loquar, principalem dependentem ab æquatione fundamentali (A) $zdx = dy$; nam ſi fiat æqualis nihilo conſtans g, quam addidimus integrando, ſtatim æquatio (E) tranſit in æquationem (A). In hoc caſu nihil refert quænam differentia $dx:q$ accepta ſit pro conſtante, cum evaneſcente g, etiam quantitas q evaneſcat.

Si æquatio (E) ulterius differentietur, non reſtituet expreſ- 3.
ſionem (D) niſi duobus in caſibus; vel ponendo $g = 0$, & procedendo ad ſecundas differentias nulla conſtante aſſumta, multiplicatis tamen terminis per quantitates æquivalentes, ut ſupra factum eſt; vel iterum differentiando, determinata prius pro conſtante fluxione $dx:q$. Utrumque patet relegendo analyſeos veſtigia. Cæterum æquatio (E) rurſus differentiata, neutra conditione impleta, formulam toto cœlo diverſam ab expreſſione (D) quam reducendam aſſumſimus, exhibet.

Idem omnino contingit, ſumto pro conſtante elemento 4.
$dy:q$; nam operationem juxta traditam methodum inſtituendo, quam brevitati conſulens omitto, deveniemus ad æquationem reductam $dx = dy : q \times \overline{mg+g}^{\frac{1}{m+1}}$ reſpondentem noſtræ formulæ (D), in quo pariter notandum; quod facta conſtante ſuperaddita $g = 0$, prodit expreſſio fundamentalis (A) $zdx = dy$.

Denique ex dictis colligi poſſe videtur, quod propoſita 5.
nuda formula differentiali ſecundi gradus (D) $z^{m+1} dx^m ddx + dy^{m+1} dz : z = dy^m ddy$, putarem ex aſſe me ſatisfeciſſe Analyſtæ

I 3 quan-

quantumvis moroſo, obſervando ad hanc expreſſionem perveniri potuiſſe, vel nulla conſtante aſſumta, quo in caſu locum invenit æquatio integralis $zdx = dy$, vel deſignando pro conſtantibus fluxiones $dx : q$; $dy : q$, & tunc ſummatorias eſſe $dy = zdx : q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}$; $dx = dy : z - dy : q \times \overline{mg + g}^{\frac{x}{m+1}}$. Adderem, æquationem unicam $zdx = dy$, quæ differentiata absque conſtantis beneficio tranſit in æquationem (D) ab aliis infinitis artificio ſupra explicato diſtingui poſſe, quia ſemper eadem manet in quacunque conſtantis ſuppoſitione, reliquæ vero variata conſtante mutationi ſunt obnoxiæ.

Supereſt ut videamus, utrum in aliis expreſſionibus, & præcipue in æquatione (F) $x^m ddx = ddy$ reſpondente problemati ab initio propoſito aſſignatæ conditiones adimpleantur ad mutationem conſtantis, & quæ rurſus differentiata, nulla conſtante aſſumta formulam (F) ſaltem terminis additis, vel ſubductis, aut valoribus ſubrogatis, ſalva æqualitate reſtituat. Fiat igitur de more $dx = qdp$, eritque propter conſtantem dp, $ddx = dqdp$. Sit iterum $dy = udp$, hoc eſt $ddy = dudp$, & ſubſtituendo $x^m dqdp = dudp$, ſeu $x^m dq = du$, & integrando $\int x^m dq + g = u$; ſed $dy = udx : q$; ergo $dy = dx : q \times \int x^m dq + gdx : q$.

Hic noto, quod facta $g = o$, & reducta ultima æquatione ad ſimpliciorem formam, videlicet $dy = \int x^m dq \times dx : q$, quilibet valor ipſius q æquationem, & curvam diverſam ſubminiſtrat, niſi fortaſſe poneretur exponens $m = o$, quod aſſumtam hypotheſin evertit. Idem dicendum ſtatuta conſtante fluxione $dy : q$, ex quibus infero, fruſtra quæri æquationem differentialem primi ordinis, quæ propoſitum præſtare queat, & formulam (F) $x^m ddx = ddy$, ſine conſtantis auxilio, reſtituere; nam ſi daretur talis æquatio, prodere ſe deberet in quacunque conſtantis ſuppoſitione, cum tamen noſtra analyſis contrarium oſtendat.

Conſtat igitur problema propoſitum, nempe curvam invenire, in qua data dignitas abſciſſæ ſit ſemper directe ut ſecunda fluxio ordinatæ, & reciproce ut ſimilis fluxio ejusdem abſciſſæ, ſolvi non poſſe, ſi curva quæſita talem proprietatem obtinere debeat, ſumtis ſecundis differentiis nulla conſtante determinata, &

& ex opposito curvas infinitas satisfacere, si modo una, modo altera constans usurpanda sit.

Non erit abs re aliud exemplum in medium afferre, & sequentem formulam (G) $x^m ddx = zddx + dz^2 + zz dz^2$ examini subjicere. Hanc sub canone nostro non comprehendi, videtur primo aspectu colligi ex eo, quod æquatio utramque indeterminatam x, z cum suis functionibus contineat: verum si fiat $zdz = dy$, nova expressio (H) $x^m ddx = ddy + dy^2$ ex hac substitutione resultans juxta regulas supra explicatas solutionem non respuit.

Inprimis designo pro constante differentiam dx, unde fit $ddx = 0$. Evanescente igitur termino $x^m ddx$, remanet $-ddy = dy^2$, vel $-ddy : dy = dy$, & integrando log. $dx : dy = y$, vel $dx : dy = 1^y$, hoc est $dx = 1^y dy$, & tandem $dx : x = dy$, quæ æquatio dat logarithmicam vulgarem. 1.

Statuo tanquam constantem alteram differentiam dy, in qua hypothesi, existente $ddy = 0$, erit $x^m ddx = dy^2$. Pono $dx = sdy + cdy$, c constans est sive affirmativa sive negativa, & s variabilis determinanda. Transeo ad secundas differentias, & sese offert æquatio $ddx = dsdy$. Hinc substituendo $x^m ds = dy$, sed $dy = dx : \overline{s+c}$; igitur $sds + cds = x^{-m} dx$, & summando omissa inutili constantis g additione $ss : 2 + cs = x^{-m+1} : -\overline{m+1}$, seu $s + c = \surd\, \overline{2x^{-m+1} : -\overline{m+1}) + cc}$: atqui $dx = \overline{s+c} \times dy$ $= dy \times \surd\, \overline{2x^{-m+1} : -\overline{m+1}) + cc}$; igitur $dx : \surd\, \overline{2x^{-m+1} : -\overline{m+1} + cc} = dy$. 2.

Quæro utrum logarithmica inventa num. 1 adhibita constante dx, locum pariter habere possit in suppositione constantis dy num. 2 usurpata. Facta constante $c = 0$, periculum facio an forte quantitas $\surd\, \overline{2x^{-m+1} : -\overline{m+1}}$ possit esse æqualis magnitudini x. Quoniam hinc inde quadrando $2x^{-m+1} : -\overline{m+1} = xx$; igitur $2x^{-m+1} = -\overline{m+1}\, xxx$, & cum eadem quantitas debeat esse, tam in coefficiente, quam in exponente binario æqualis, sequitur ad æqualitatem non perveniri, nisi ponendo $-m+1 = 2$, quo in casu determinatur valor exponentis $m = -1$. 3.

in.

4. In formula (H) $x^m ddx = ddy + dy^2$, limitando, ut dictum est, valorem exponentis $m = -1$, tunc ad æquationem $x^{-1} ddx = ddy + dy^2$ pervenire possumus nulla assumta constante, ejusque summatoria in hac hypothesi est æquatio differentialis ad logisticam $dx:x = dy$; nam ascendendo ad secundas differentias absque constantis auxilio, habebimus $ddx:x - dx^2:xx = ddy$, sed $dx^2:xx = dy^2$; ergo $ddx:x = ddy + dy^2$.

5. Quod si valor ipsius m non sit æqualis quantitati negativæ -1, ad expressionem (H) nullo modo pervenire conceditur, nisi aliqua fluxio tanquam constans determinetur.

6. Procedendo autem generaliter, ut supra factum est, repeto æquationem (H) $x^m ddx = ddy + dy^2$. Sumo pro constante elementum $dx:q = dp$, & statuo pariter $dy = udp$, ut obtineam rursus differentiando $ddx = dqdp$; $ddy = dudp$, & substituendo $x^m dqdp = dudp + dy^2 = dudp + udydp$, & dividendo per dp, $x^m dq = du + udy$; sed $dy = udp = udx:q$; igitur $x^m dq = du + uudx:q$.

7. Methodus generaliter separandi variabiles in hac expressione, etiamsi quantitas q detur quocunque modo per functiones solius ignotæ x, pro desperata habenda est. Moneo tamen, quod si fiat exponens $m = -1$, simplicior indeterminatæ u valor prodit æqualis fractioni $q:x$; nam loco ipsius u hoc valore subrogato, omnes termini in æquatione se mutuo destruunt. Hinc collocata in æquatione subsidiaria $dy = adp = udx:q$ hoc valore, redit æquatio ad logarithmicam $dy = dx:x$, quæcunque fuerit constans assumta per magnitudinem $dx:q$ expressa.

Denique manifestum est, nostrum operandi progressum in maximam difficultatem separationis indeterminatarum postremo desinere. Hanc spartam olim exornandam suscepi, specimenque aliquod in Diario Italico exhibui: sed aut ego fallor, aut negotium tam subtile, tam arduum, ex quo potissimum pendet calculi infinitorum optata perfectio, non nisi conjunctis viribus promovendum est. Ut igitur ad hanc inquisitionem profundioris analysis & Geometriæ cultores excitem, sequens problema propono.

In

In ſuperiori formula $x^m dq = du + uudx : q$, dato ad libitum exponente m, ſtatuatur quantitas $q = x^n$. Peto qua ratione determinandi ſint valores alterius exponentis n, ut ſuccedat indeterminatarum ſeparatio, & æquationis conſtructio per ſolas quadraturas.

L. Euler

DE RESOLUTIONE AEQUATIONIS

$dy + a\, yydx = bx^m\, dx$

Novi Commentarii Academiae Scientiarum Petropolitanae,
1764

From Leonhardi Euleri's Opera Omnia, Volumen Prius, *Basileae 1936,*
B. G. Teubner Lipsiae et Berolini, pages 403-420.

DE RESOLUTIONE AEQUATIONIS
$dy + ayydx = bx^m dx$

Commentatio 284 indicis ENESTROEMIANI

Novi Commentarii academiae scientiarum Petropolitanae 9 (1762/3, 1764) p. 154—169

Summarium ibidem p. 18—21

SUMMARIUM

Aequatio haec, iam dudum a Comite RICCATI Geometris proposita, tanto studio a summis ingeniis est pertractata, ut vix quicquam novi circa eius resolutionem proferri posse videatur. Statim quidem infiniti valores pro exponente m assumendi sunt observati, quibus integrale exhibere liceat, qui valores hac serie progrediuntur: $0, -4, -\frac{4}{3}, -\frac{8}{3}, -\frac{8}{5}, -\frac{12}{5}, -\frac{12}{7}, -\frac{16}{7}, -\frac{16}{9}$ etc., ac methodus, qua hi casus sunt evoluti, ita erat comparata, ut ex cognito cuiusque casus integrali integrale sequentis definiretur, neque adeo casuum posteriorum integralia exhiberi possent, nisi iam omnes antecedentes fuerint expediti. In hac autem dissertatione id praestatur, ut unica operatione omnium illorum casuum integralia simul eruantur, indeque statim vel centesimi casus integrale assignari possit. Methodus, qua hoc commodi est assecutus, omnino est singularis, dum primo aequationem propositam, ope certae substitutionis, in aliam, quae adeo differentialia secundi gradus involvit, transformat, eamque deinceps per seriem infinitam integrat, quae autem series ita est comparata, ut supra memoratis casibus alicubi abrumpatur expressionemque finitam suppeditet, unde integrale quaesitum facillime colligatur. Verum tamen omnia haec integralia nonnisi sunt particularia, neque totam vim aequationis differentialis propositae exhauriunt, deinde etiam, quoties quantitas b est negativa, imaginariis ita inquinantur, ut omni plane usu destituantur. Utrique incommodo Cel. Auctor ita medetur, ut primo methodum exponat, ex cognito huiusmodi aequationum integrali quopiam particulari integrale completum eliciendi, quod si quantitas b fuerit positiva, quantitates exponentiales implicat: deinde vero ostendit, quomodo istae quantitates exponentiales, quae, existente b negativo, fiunt imaginariae, per tangentes arcuum circularium realiter exprimi queant. Denique cum methodus illa, ex integrali particulari

completum eliciendi, certam quandam integrationem exigat, quae moram facessere queat, etiam huic incommodo occurrit, dum observat, pro quovis casu primam evolutionem non unum, sed adeo duo integralia particularia praebere, quoniam ibi formula radicalis $\sqrt{b}$ ingreditur, quam aeque negative, ac positive, accipere licet. Alia igitur methodo utitur, cuius ope ex cognitis duobus integralibus particularibus integrale completum, sine ulla nova integratione, concludi queat. Quod cum ab eo, quod priori methodo erat erutum, discrepare nequeat, ex utriusque collatione integrationem priori implicatam efficere licet, unde postremo hanc integrationem maxime memorabilem deducit, quod sit

$$\int \frac{e^{\frac{2ac}{n}x^n}\,dx}{uu} = \frac{Ce^{\frac{2ac}{n}x^n}z - u}{Cu\left(2acx^{n-1}uz + \frac{udz}{dx} - \frac{zdu}{dx}\right)},$$

ubi quantitates z et u per x ita definiuntur, ut sit:

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)(9nn-1)}{8n\cdot 16na^2c^2}x^{\frac{-5n+1}{2}} + \text{etc.}$$

$$u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)(9nn-1)}{8n\cdot 16na^2c^2}x^{\frac{-5n+1}{2}} - \text{etc.}$$

Cum igitur hae formae z et u adeo in infinitum excurrere queant, eo magis est mirandum, quod formulae $e^{\frac{2ac}{n}x^n}\frac{dx}{uu}$ integrale, idque per expressionem satis simplicem, exhiberi possit. Tum vero etiam hoc consuetae integralium formae adversari videtur, quod quantitas constans arbitraria C, per integrationem ingressa, quae alioquin nude adiicitur, hic ipsi formae integrali sit implicata. Quod singulare phaenomenon si attentius perpendatur, mox patebit, integrationem illam veritati consentaneam esse non posse, nisi denominatoris pars

$$2acx^{n-1}uz + \frac{udz - zdu}{dx}$$

fuerit quantitas constans, puta A; tum enim istud integrale in formam naturalem abit:

$$e^{\frac{2ac}{n}x^n}\frac{z}{Au} - \frac{1}{AC}.$$

Num autem res ita se habeat, hoc modo explicari potest: Quoniam quantitates z et u per series exprimuntur, easque ipsas, quae initio ex evolutione aequationis differentialis secundi gradus sunt eruta, vicissim patet, eas ita pendere ab x, ut sit:

$$ddz + 2acx^{n-1}dxdz + (n-1)acx^{n-2}zdx^2 = 0$$

et

$$ddu - 2acx^{n-1}dxdu - (n-1)acx^{n-2}udx^2 = 0.$$

Nunc prior aequatio per u, posterior vero per z, multiplicetur, ac productorum differentia dabit

$$uddz - zddu + 2acx^{n-1}dx(udz + zdu) + 2(n-1)acx^{n-2}uzdx^2 = 0,$$

cuius integrale manifesto est

$$udz - zdu + 2\,acx^{n-1}uzdx = Adx.$$

Cum autem, facto $ac = \varpi$, fiat $u = z = x^{\frac{-n+1}{2}}$ et $uz = x^{-n+1}$, evidens est, statui debere $A = 2\,ac$, sicque integratio superior abit in hanc formam:

$$\int e^{\frac{2ac}{n}x^n}\frac{dz}{uu} = e^{\frac{2ac}{n}x^n}\frac{z}{2acu} - \text{Const.},$$

quae non solum principiis est conformis, sed etiam, facta differentiatione, ob

$$udz - zdu = 2\,acdx\,(1 - x^{n-1}ux)$$

eius veritas egregie confirmatur. Hinc autem iam aequationis

$$dy + ayydx = accx^m dx,$$

posito $m = 2n - 2$, et quantitatis z valore per superiorem seriem expresso, integrale multo succinctius ita exhiberi poterit, ut sit:

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{2\,Cce^{\frac{-2ac}{n}x^n}}{z(z - Ce^{\frac{-2ac}{n}x^n}u)}$$

seu

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{2c}{z(De^{\frac{2ac}{n}x^n}z - u)}$$

ubi $D\left[= \frac{1}{C}\right]$ est illa constans arbitraria per integrationem iniecta ad integrale completum constituendum.

PROBLEMA 1

1. Invenire numeros loco exponentis indefiniti m substituendos, ut valor ipsius y algebraice per x definiri queat.

SOLUTIO[1])

Ponatur

$$y = cx^{n-1} + \frac{dz}{azdx},$$

ac posito dx constante, erit

$$dy = (n-1)cx^{n-2}dx + \frac{ddz}{azdx} - \frac{dz^2}{azzdx}.$$

1) Cf. L. EULERI Commentationem 95 huius voluminis p. 162 et *Institutiones calculi integralis*, vol. II, § 929—966. Petr. 1769 = *LEONHARDI EULERI Opera omnia*, I 12, p. 147—176. H. D.

Cum vero sit

$$yy = ccx^{2n-2} + \frac{2cx^{n-1}dz}{azdx} + \frac{dz^2}{a^2z^2dx^2},$$

facta substitutione transibit aequatio proposita in hanc:

$$\frac{ddz}{azdx} + (n-1)cx^{n-2}dx + accx^{2n-2}dx + \frac{2cx^{n-1}dz}{z} = bx^m dx.$$

Fiat $m = 2n - 2$ et $b = acc$, habebiturque

$$ddz + (n-1)\,acx^{n-2}zdx^2 + 2\,acx^{n-1}\,dxdz = 0,$$

quae ergo resultat ex hac aequatione propositae aequivalente

$$dy + ayydx = accx^{2n-2}dx$$

facta substitutione

$$y = cx^{n-1} + \frac{dz}{azdx}.$$

Fingatur iam haec aequatio:

$$z = Ax^{\frac{-n+1}{2}} + Bx^{\frac{-3n+1}{2}} + Cx^{\frac{-5n+1}{2}} + Dx^{\frac{-7n+1}{2}} + \text{etc.}$$

eritque differentiando:

$$\frac{dz}{dx} = -\frac{(n-1)}{2}Ax^{\frac{-n-1}{2}} - \frac{(3n-1)}{2}Bx^{\frac{-3n-1}{2}} - \frac{(5n-1)}{2}Cx^{\frac{-5n-1}{2}} - \text{etc.}$$

$$\frac{ddz}{dx^2} = +\frac{(nn-1)}{4}Ax^{\frac{-n-3}{2}} + \frac{(9nn-1)}{4}Bx^{\frac{-3n-3}{2}} + \frac{(25nn-1)}{4}Cx^{\frac{-5n-3}{2}} + \text{etc.}$$

Cum vero ex superiori aequatione per dx^2 divisa sit:

$$\frac{ddz}{dx^2} + \frac{2acx^{n-1}dz}{dx} + (n-1)acx^{n-2}z = 0,$$

si series assumta substituatur, prodibit sequens aequatio:

$$0 = \left\{ \begin{array}{l}
\qquad + \frac{(nn-1)}{4}\,Ax^{\frac{-n-3}{2}} + \frac{(9nn-1)}{4}\,Bx^{\frac{-3n-3}{2}} \\
\qquad + \frac{(25nn-1)}{4}\,Cx^{\frac{-5n-3}{2}} + \frac{(49nn-1)}{4}\,Dx^{\frac{-7n-3}{2}} + \text{etc.} \\
-(n-1)acAx^{\frac{n-3}{2}} - (3n-1)acBx^{\frac{-n-3}{2}} - (5n-1)acCx^{\frac{-3n-3}{2}} \\
\qquad - (7n-1)acDx^{\frac{-5n-3}{2}} - (9n-1)acEx^{\frac{-7n-3}{2}} - \text{etc.} \\
+(n-1)acAx^{\frac{n-3}{2}} + (n-1)acBx^{\frac{-n-3}{2}} + (n-1)acCx^{\frac{-3n-3}{2}} \\
\qquad + (n-1)acDx^{\frac{-5n-3}{2}} + (n-1)acEx^{\frac{-7n-3}{2}} + \text{etc.}
\end{array} \right.$$

Ponantur termini homogenei iunctim sumti nihilo aequales, ut determinentur coefficientes A, B, C, D, E etc., eritque

$$B = \frac{(nn-1)}{2n} \cdot \frac{A}{4ac} = \frac{(nn-1)}{2} \cdot \frac{A}{4nac}$$

$$C = \frac{(9nn-1)}{4n} \cdot \frac{B}{4ac} = \frac{(nn-1)}{2}\,\frac{(9nn-1)}{4} \cdot \frac{A}{4^2 n^2 a^2 c^2}$$

$$D = \frac{(25nn-1)}{6n} \cdot \frac{C}{4ac} = \frac{(nn-1)}{2}\,\frac{(9nn-1)}{4}\,\frac{(25nn-1)}{6} \cdot \frac{A}{4^3 n^3 a^3 c^3}$$

$$E = \frac{(49nn-1)}{8n} \cdot \frac{D}{4ac} = \frac{(nn-1)}{2}\,\frac{(9nn-1)}{4}\,\frac{(25nn-1)}{6}\,\frac{(49nn-1)}{8} \cdot \frac{A}{4^4 n^4 a^4 c^4}$$

etc.

Determinabitur ergo z per x sequenti modo:

$$z = Ax^{\frac{-n+1}{2}} + \frac{(nn-1)}{8}\,\frac{A}{nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8}\,\frac{(9nn-1)}{16}\,\frac{A}{n^2a^2c^2}x^{\frac{-5n+1}{2}}$$
$$+ \frac{(nn-1)}{8}\,\frac{(9nn-1)}{16}\,\frac{(25nn-1)}{24}\,\frac{A}{n^3a^3c^3}x^{\frac{-7n+1}{2}} + \text{etc.}$$

Valore hoc substituto resultabit valor quaesitus: $y = cx^{n-1}$

$$-\frac{1}{a}\left\{\frac{\frac{(n-1)}{2}Ax^{\frac{-n-1}{2}} + \frac{(3n-1)}{2}\frac{(nn-1)}{8}\frac{A}{nac}x^{\frac{-3n-1}{2}} + \frac{(5n-1)}{2}\frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{A}{n^2a^2c^2}x^{\frac{-5n-1}{2}} + \text{etc.}}{Ax^{\frac{-n+1}{2}} + \frac{(nn-1)}{8}\frac{A}{nac}x^{\frac{-3n+1}{2}} + \frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{A}{n^2a^2c^2}x^{\frac{-5n+1}{2}} + \text{etc.}}\right\}$$

sive numeratore ac denominatore per $Ax^{\frac{-n-1}{2}}$ diviso: $y = cx^{n-1}$

$$\frac{1}{ax}\left\{\frac{\frac{(n-1)}{2} + \frac{(3n-1)}{2}\frac{(nn-1)}{8}\frac{x^{-n}}{nac} + \frac{(5n-1)}{2}\frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{x^{-2n}}{n^2a^2c^2} + \frac{(7n-1)}{2}\frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{(25nn-1)}{24}\frac{x^{-3n}}{n^3a^3c^3} + \text{etc.}}{1 + \frac{(nn-1)}{8}\frac{x^{-n}}{nac} + \frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{x^{-2n}}{n^2a^2c^2} + \frac{(nn-1)}{8}\frac{(9nn-1)}{16}\frac{(25nn-1)}{24}\frac{x^{-3n}}{n^3a^3c^3} + \text{etc.}}\right\}$$

Haec ergo expressio generaliter in infinitum excurrens fit finita, si fuerit

$$(2i+1)^2 nn - 1 = 0,$$

denotante i numerum quemcunque integrum, hoc est, si fuerit

$$n = \frac{\pm 1}{2i+1} \quad \text{et} \quad m = 2n - 2 = \frac{-4i - 2 \pm 2}{2i+1}.$$

Huius ergo aequationis, quoties i fuerit numerus integer:

$$dy + ayydx = accx^{\frac{-4i-2\pm 2}{2i+1}}dx$$

integrale semper in terminis finitis poterit exhiberi, seu valor ipsius y per x algebraice exponi.

Sit primo $n = \frac{+1}{2i+1}$, ut sit $m = 2n - 2 = \frac{-4i}{2i+1}$, erit huius aequationis:

$$dy + ayydx = accx^{\frac{-4i}{2i+1}}dx$$

integrale in terminis algebraicis expressum:

$$ayx = acx^{\frac{1}{2i+1}}$$

$$+ \frac{\frac{i}{2i+1} - \frac{i(i^2-1)}{2(2i+1)^2}\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i^2-4)}{2\cdot 4(2i+1)^3}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i^2-9)}{2\cdot 4\cdot 6(2i+1)^4}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}{1 - \frac{i(i+1)}{2(2i+1)}\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

seu facta ad communem denominatorem reductione erit:

$$ayx = \frac{acx^{\frac{+1}{2i+1}} - \frac{i(i-1)}{2(2i+1)} + \frac{i(i^2-1)(i-2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-1}{2i+1}}}{ac} - \frac{i(i^2-1)(i^2-4)(i-3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} + \text{etc.}}{1 - \frac{i(i+1)}{2(2i+1)}\frac{x^{\frac{-1}{2i+1}}}{ac} + \frac{i(i^2-1)(i+2)}{2\cdot 4(2i+1)^2}\frac{x^{\frac{-2}{2i+1}}}{a^2c^2} - \frac{i(i^2-1)(i^2-4)(i+3)}{2\cdot 4\cdot 6(2i+1)^3}\frac{x^{\frac{-3}{2i+1}}}{a^3c^3} + \text{etc.}}$$

Sit deinde $n = \frac{-1}{2i+1}$, ut sit $m = \frac{-4i-4}{2i+1}$, erit huius aequationis

$$dy + ayydx = accx^{\frac{-4i-4}{2i+1}}dx$$

integrale in terminis algebraicis expressum:

$$ayx = acx^{\frac{-1}{2i+1}}$$

$$+\frac{\dfrac{i+1}{2i+1}+\dfrac{i(i+1)(i+2)}{2(2i+1)^2}\dfrac{x^{\frac{1}{2i+1}}}{ac}+\dfrac{i(i^2-1)(i+2)(i+3)}{2\cdot4(2i+1)^3}\dfrac{x^{\frac{2}{2i+1}}}{a^2c^2}+\dfrac{i(i^2-1)(i^2-4)(i+3)(i+4)}{2\cdot4\cdot6(2i+1)^4}\dfrac{x^{\frac{3}{2i+1}}}{a^3c^3}+\text{etc.}}{1+\dfrac{i(i+1)}{2(2i+1)}\dfrac{x^{\frac{1}{2i+1}}}{ac}+\dfrac{i(i^2-1)(i+2)}{2\cdot4(2i+1)^2}\dfrac{x^{\frac{2}{2i+1}}}{a^2c^2}+\dfrac{i(i^2-1)(i^2-4)(i+3)}{2\cdot4\cdot6(2i+1)^3}\dfrac{x^{\frac{3}{2i+1}}}{a^3c^3}+\text{etc.}}$$

seu facta ad communem denominatorem reductione, erit $ayx =$

$$\frac{acx^{\frac{-1}{2i+1}}+\dfrac{(i+1)(i+2)}{2(2i+1)}+\dfrac{i(i+1)(i+2)(i+3)}{2\cdot4(2i+1)^2}\dfrac{x^{\frac{1}{2i+1}}}{ac}+\dfrac{i(i^2-1)(i+2)(i+3)(i+4)}{2\cdot4\cdot6(2i+1)^3}\dfrac{x^{\frac{2}{2i+1}}}{a^2c^2}+\text{etc.}}{1+\dfrac{i(i+1)}{2(2i+1)}\dfrac{x^{\frac{1}{2i+1}}}{ac}+\dfrac{i(i^2-1)(i+2)}{2\cdot4(2i+1)^2}\dfrac{x^{\frac{2}{2i+1}}}{a^2c^2}+\dfrac{i(i^2-1)(i^2-4)(i+3)}{2\cdot4\cdot6(2i+1)^3}\dfrac{x^{\frac{3}{2i+1}}}{a^3c^3}+\text{etc.}}$$

Quotiescunque igitur fuerit i numerus integer, toties huius aequationis:

$$dy + ayydx = accx^{\frac{-4i-2\pm2}{2i+1}}dx$$

integrale in terminis algebraicis potest exprimi. Q. E. I.

COROLLARIUM 1

2. Aequatio ergo proposita

$$dy + ayydx = accx^m dx$$

integrationem algebraicam admittit, si fuerit exponens m vel terminus huius seriei:

$$-0,\ -\frac{4}{3},\ -\frac{8}{5},\ -\frac{12}{7},\ -\frac{16}{9},\ -\frac{20}{11},\ -\frac{24}{13},\ \text{etc.}$$

vel si fuerit m terminus ex hac fractionum serie:

$$-\frac{4}{1},\ -\frac{8}{3},\ -\frac{12}{5},\ -\frac{16}{7},\ -\frac{20}{9},\ -\frac{24}{11},\ -\frac{28}{13},\ \text{etc.}$$

COROLLARIUM 2

3. Substituamus in priori integrabilitatis classe loco i successive numeros 0, 1, 2, 3, 4 etc. atque reperietur, ut sequitur.

Si $i = 0$, huius aequationis:

$$\text{I. } dy + ayydx = accdx$$

integrale erit:

$$ayx = acx \quad \text{sive} \quad y = c.$$

Si $i = 1$, huius aequationis:

$$\text{II. } dy + ayydx = accx^{-\frac{4}{3}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{3}}}{1 - \frac{1 \cdot 2}{2 \cdot 3} \cdot \frac{x^{-\frac{1}{3}}}{ac}} \quad \text{seu} \quad y = \frac{cx^{-\frac{2}{3}}}{1 - \frac{1}{3}\frac{x^{-\frac{1}{3}}}{ac}} = \frac{3acc}{3acx^{\frac{2}{3}} - x^{\frac{1}{3}}}.$$

Si $i = 2$, huius aequationis:

$$\text{III. } dy + ayydx = accx^{-\frac{8}{5}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{5}} - \frac{2 \cdot 1}{2 \cdot 5}}{1 - \frac{2 \cdot 3}{2 \cdot 5} \cdot \frac{x^{-\frac{1}{5}}}{ac} + \frac{2 \cdot 3 \cdot 4}{2 \cdot 4 \cdot 5^2} \cdot \frac{x^{-\frac{2}{5}}}{a^2c^2}} = \frac{acx^{\frac{1}{5}} - \frac{1}{5}}{1 - \frac{3x^{-\frac{1}{5}}}{5ac} + \frac{3x^{-\frac{2}{5}}}{5^2a^2c^2}}.$$

Si $i = 3$, huius aequationis:

$$\text{IV. } dy + ayydx = accx^{-\frac{12}{7}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{7}} - \frac{3 \cdot 2}{2 \cdot 7} + \frac{3 \cdot 2 \cdot 1 \cdot 4}{2 \cdot 4 \cdot 7^2} \cdot \frac{x^{-\frac{1}{7}}}{ac}}{1 - \frac{3 \cdot 4}{2 \cdot 7} \cdot \frac{x^{-\frac{1}{7}}}{ac} + \frac{3 \cdot 4 \cdot 5 \cdot 2}{2 \cdot 4 \cdot 7^2} \cdot \frac{x^{-\frac{2}{7}}}{a^2c^2} - \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdot 2}{2 \cdot 4 \cdot 6 \cdot 7^3} \cdot \frac{x^{-\frac{3}{7}}}{a^3c^3}}$$

sive

$$ayx = \frac{acx^{\frac{1}{7}} - \frac{3}{7} + \frac{3 \cdot 1}{7^2} \cdot \frac{x^{-\frac{1}{7}}}{ac}}{1 - \frac{6}{7} \cdot \frac{x^{-\frac{1}{7}}}{ac} + \frac{3 \cdot 5}{7^2} \cdot \frac{x^{-\frac{2}{7}}}{a^2c^2} - \frac{1 \cdot 3 \cdot 5}{7^3} \cdot \frac{x^{-\frac{3}{7}}}{a^3c^3}}.$$

Si $i = 4$, huius aequationis:

$$\text{V. } dy + ayydx = accx^{-\frac{16}{9}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{9}} - \frac{4\cdot 3}{2\cdot 9} + \frac{4\cdot 3\cdot 2\cdot 5}{2\cdot 4\cdot 9^2}\cdot\frac{x^{-\frac{1}{9}}}{ac} - \frac{4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 9^3}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2}}{1 - \frac{4\cdot 5}{2\cdot 9}\cdot\frac{x^{-\frac{1}{9}}}{ac} + \frac{4\cdot 5\cdot 6\cdot 3}{2\cdot 4\cdot 9^2}\cdot\frac{x^{-\frac{2}{9}}}{a^2c^2} - \frac{4\cdot 5\cdot 6\cdot 7\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 9^3}\cdot\frac{x^{-\frac{3}{9}}}{a^3c^3} + \frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 6\cdot 8\cdot 9^4}\cdot\frac{x^{-\frac{4}{9}}}{a^4c^4}}.$$

Si $i = 5$, huius aequationis

$$\text{VI. } dy + ayydx = accx^{-\frac{20}{11}}dx$$

integrale erit:

$$ayx = \frac{acx^{\frac{1}{11}} - \frac{5\cdot 4}{2\cdot 11} + \frac{5\cdot 4\cdot 3\cdot 6}{2\cdot 4\cdot 11^2}\frac{x^{-\frac{1}{11}}}{ac} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 6\cdot 7}{2\cdot 4\cdot 6\cdot 11^3}\frac{x^{-\frac{2}{11}}}{a^2c^2} + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6\cdot 8\cdot 11^4}\frac{x^{-\frac{3}{11}}}{a^3c^3}}{1 - \frac{5\cdot 6}{2\cdot 11}\frac{x^{-\frac{1}{11}}}{ac} + \frac{5\cdot 6\cdot 7\cdot 4}{2\cdot 4\cdot 11^2}\frac{x^{-\frac{2}{11}}}{a^2c^2} - \frac{5\cdot 6\cdot 7\cdot 8\cdot 4\cdot 3}{2\cdot 4\cdot 6\cdot 11^3}\frac{x^{-\frac{3}{11}}}{a^3c^3} + \frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 11^4}\frac{x^{-\frac{4}{11}}}{a^4c^4} - \frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 11^5}\frac{x^{-\frac{5}{11}}}{a^5c^5}}.$$

COROLLARIUM 3

4. In posteriori integrabilitatis ordine substituamus pariter loco i numeros 0, 1, 2, 3, 4 etc. ac reperietur, ut sequitur.

Si $i = 0$, huius aequationis:

$$\text{I. } dy + ayydx = accx^{-4}dx$$

integrale erit:

$$ayx = \frac{acx^{-1} + \frac{1\cdot 2}{2\cdot 1}}{1} = 1 + \frac{ac}{x} \text{ seu } y = \frac{1}{ax} + \frac{c}{xx}.$$

Si $i = 1$, huius aequationis:

$$\text{II. } dy + ayydx = accx^{-\frac{8}{3}}dx$$

integrale erit:

$$ayx = \frac{acx^{-\frac{1}{3}} + \frac{2\cdot 3}{2\cdot 3} + \frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 4\cdot 3^2}\frac{x^{\frac{1}{3}}}{ac}}{1 + \frac{1\cdot 2}{2\cdot 3}\cdot\frac{x^{\frac{1}{3}}}{ac}} = \frac{acx^{-\frac{1}{3}} + 1 + \frac{x^{\frac{1}{3}}}{3ac}}{1 + \frac{x^{\frac{1}{3}}}{3ac}}.$$

Si $i = 2$, huius aequationis:

$$\text{III.}\quad dy + ayydx = accx^{-\frac{12}{5}}dx$$

integrale erit:

$$ayx = \frac{acx^{-\frac{1}{5}} + \frac{3\cdot 4}{2\cdot 5} + \frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4\cdot 5^2}\cdot\frac{x^{\frac{1}{5}}}{ac} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 5^3}\cdot\frac{x^{\frac{2}{5}}}{a^2c^2}}{1 + \frac{2\cdot 3}{2\cdot 5}\cdot\frac{x^{\frac{1}{5}}}{ac} + \frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 4\cdot 5^2}\cdot\frac{x^{\frac{2}{5}}}{a^2c^2}}.$$

Si $i = 3$, huius aequationis:

$$\text{IV.}\quad dy + ayydx = accx^{-\frac{16}{7}}dx$$

integrale erit:

$$ayx = \frac{acx^{-\frac{1}{7}} + \frac{4\cdot 5}{2\cdot 7} + \frac{3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 7^2}\cdot\frac{x^{\frac{1}{7}}}{ac} + \frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{2\cdot 4\cdot 6\cdot 7^3}\cdot\frac{x^{\frac{2}{7}}}{a^2c^2} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6\cdot 8\cdot 7^4}\cdot\frac{x^{\frac{3}{7}}}{a^3c^3}}{1 + \frac{3\cdot 4}{2\cdot 7}\cdot\frac{x^{\frac{1}{7}}}{ac} + \frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4\cdot 7^2}\cdot\frac{x^{\frac{2}{7}}}{a^2c^2} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6\cdot 7^3}\cdot\frac{x^{\frac{3}{7}}}{a^3c^3}}.$$

Atque ex his casibus analogia patet, cuius ope omnium casuum, qui quidem integrationem admittunt, integralia algebraica expedite formari poterunt.

SCHOLION

5. De his integralibus autem probe notandum est, ea non esse completa, neque ideo aeque late patere, ac aequationem differentialem; id quod vel ex primo casu

$$dy + ayydx = accdx$$

patet, cui etsi satisfacit $y = c$, tamen facile intelligitur, logarithmos insuper in ea comprehendi. Manifestum autem hoc est quoque hinc, quod in his integralibus non contineatur nova constans arbitraria, quae in differentiali non inerat; in quo criterium integrationis completae versatur. Caeterum vero hinc duplicia integralia cuiusvis casus obtinentur, eo quod c tam affirmative, quam negative, accipere licet, aequatione differentiali, quae tantum cc continet, non mutata.

PROBLEMA 2

6. Invento ope praecedentis methodi integrali particulari pro casibus assignatis aequationis $dy + ayydx = accx^m dx$, invenire integrale completum pro iisdem casibus[1]).

1) Vide notam p. 405. H. D.

SOLUTIO

Posito $m = 2n - 2$, integrale particulare aequationis propositae inventum est esse $ayx = acx^n$

$$- \frac{\frac{(n-1)}{2} + \frac{(3n-1)}{2}\frac{(nn-1)}{8n}\frac{x^{-n}}{ac} + \frac{(5n-1)}{2}\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{x^{-2n}}{a^2c^2} + \frac{(7n-1)}{2}\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\frac{x^{-3n}}{a^3c^3} + \text{etc.}}{1 + \frac{(nn-1)}{8n}\frac{x^{-n}}{ac} + \frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{x^{-2n}}{a^2c^2} + \frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\frac{x^{-3n}}{a^3c^3} + \text{etc.}}$$

cuius loco scribamus brevitatis gratia $y = P$. Cum igitur P sit eiusmodi valor, per variabilem x datus, qui satisfaciat aequationi

$$dy + ayydx = accx^{2n-2}dx,$$

erit utique

$$dP + aP^2dx = accx^{2n-2}dx.$$

Ponamus iam integrale completum aequationis propositae

$$dy + ayydx = accx^{2n-2}dx$$

esse $y = P + v$, quo valore loco y substituto habebimus hanc aequationem

$$dP + dv + aP^2dx + 2aPvdx + avvdx = accx^{2n-2}dx.$$

Cum vero sit

$$dP + aP^2dx = accx^{2n-2}dx,$$

erit

$$dv + 2aPvdx + avvdx = 0.$$

Sit $v = \frac{1}{u}$, erit

$$du - 2aPudx = adx,$$

quae multiplicata per $e^{-2a\int Pdx}$ denotante e numerum, cuius logarithmus hyperbolicus est $= 1$, fit integrabilis; erit scilicet aequationis

$$e^{-2a\int Pdx}(du - 2aPudx) = e^{-2a\int Pdx}adx$$

integrale

$$e^{-2a\int Pdx}u = \int e^{-2a\int Pdx}adx;$$

ideoque

$$u = e^{2a\int Pdx}\int e^{-2a\int Pdx}adx.$$

Quo valore, cum sit $v = \frac{1}{u}$, substituto, erit integrale completum aequationis propositae

$$y = P + \frac{e^{-2a\int Pdx}}{\int e^{-2a\int Pdx} adx}.$$

At ex problemate primo est valor ipsius y particularis, quem hic ponimus

$$P = cx^{n-1} + \frac{dz}{azdx};$$

existente

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)x^{\frac{-3n+1}{2}}}{8n\quad ac} + \frac{(nn-1)(9nn-1)x^{\frac{-5n+1}{2}}}{8n\cdot 16n\quad a^2c^2} + \frac{(nn-1)(9nn-1)(25nn-1)x^{\frac{-7n+1}{2}}}{8n\cdot 16n\cdot 24n\quad a^3c^3} + \text{etc.}$$

Hinc erit

$$\int Pdx = \frac{cx^n}{n} + \frac{1}{a}lz \quad \text{et} \quad e^{-2a\int Pdx} = e^{\frac{-2acx^n}{n}} : zz.$$

Quo valore substituto habebitur integrale completum:

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{e^{\frac{-2acx^n}{n}}}{zz\int e^{-\frac{2acx^n}{n}} adx : zz}. \qquad \text{Q. E. I.}$$

ALITER

Quemadmodum hac ratione ex uno integrali particulari invenitur integrale completum, ita ex duobus integralibus particularibus expeditius integrale completum indagabitur, neque in hoc modo pervenitur ad formulam integralem, cuiusmodi est ea $\int e^{\frac{-2acx^n}{n}} adx : zz$, quae integrali completo, quod invenimus, involvitur. Cum enim aequatio

$$dy + ayydx = accx^{2n-2}dx$$

maneat invariata, sive c affirmative, sive negative accipiatur, habemus utique duo integralia particularia, quorum prius est

$$y = P = cx^{n-1} + \frac{dz}{azdx},$$

existente

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8n}\cdot\frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)}{8n}\,\frac{(9nn-1)}{16n}\cdot\frac{x^{\frac{-5n+1}{2}}}{a^2c^2} + \text{etc.}$$

Posterius vero simili modo investigandum erit

$$y = Q = -cx^{n-1} + \frac{du}{audx};$$

fietque

$$u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)}{8n} \frac{(9nn-1)}{16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} - \text{etc.},$$

qui duo valores z et u tantum signis inter se differunt. Erit ergo tam

$$dP + aP^2dx = accx^{2n-2}dx,$$

quam

$$dQ + aQ^2dx = accx^{2n-2}dx.$$

Ponamus iam[1])

$$R = \frac{P-y}{Q-y},$$

quae aequatio sit integralis completa propositae differentialis; quam formam ideo assumimus, quia in ea utraque particularium $y = P$ et $y = Q$ continetur, illa nempe si fiat $R = 0$, haec si $R = \infty$. Fiet ergo $QR - Ry = P - y$ hincque

$$y = \frac{QR - P}{R-1},$$

quae dat

$$dy = \frac{RRdQ - QdR - RdQ - RdP + dP + PdR}{(R-1)^2},$$

substituantur hic valores supra inventi

$$dP = -aP^2dx + accx^{2n-2}dx$$

et

$$dQ = -aQ^2dx + accx^{2n-2}dx,$$

eritque

$$dy = accx^{2n-2}dx + \frac{aP^2dx}{R-1} - \frac{aQ^2Rdx}{R-1} + \frac{(P-Q)dR}{(R-1)^2} = -a\frac{(QR-P)^2dx}{(R-1)^2} + accx^{2n-2}dx.$$

Ex hac aequatione resultat haec

$$(P-Q)\,dR = -aRdx\,(P-Q)^2,$$

quae divisa per $R\,(P-Q)$ dat

1) Cf. L. EULERI Commentationem 269; vide p. 389 huius voluminis. H. D.

$$\frac{dR}{R}=a(Q-P)dx=-2acx^{n-1}dx+\frac{du}{u}-\frac{dz}{z}.$$

Haec iam aequatio integrabilis existit, eritque integrale

$$lR-lC=-\frac{2acx^n}{n}+lu-lz.$$

Cum vero sit $R=\frac{P-y}{Q-y}$, erit

$$\frac{P-y}{Q-y}=\frac{(acx^{n-1}zdx+dz-ayzdx):z}{(-acx^{n-1}udx+du-ayudx):u}=\frac{Ce^{\frac{-2acx^n}{n}}u}{z}.$$

Hinc ita, quia valores ipsarum u et z per x constant, habebitur aequatio integralis completa

$$Ce^{\frac{-2acx^n}{n}}=\frac{dz+acx^{n-1}zdx-ayzdx}{du-acx^{n-1}udx-ayudx}=\frac{(P-y)z}{(Q-y)u}. \qquad \text{Q. E. I.}$$

COROLLARIUM 1

7. Valor particularis, quem supra pro y invenimus, ita erat comparatus, ut esset

$$y=cx^{n-1}-\frac{(K+L)}{ax(M+N)};$$

existente

$$K=\frac{(n-1)}{2}+\frac{(5n-1)}{2}\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\cdot\frac{x^{-2n}}{a^2c^2}+\frac{(9n-1)}{2}\frac{(n^2-1)}{8n}\frac{(9n^2-1)}{16n}\frac{(25n^2-1)}{24n}\frac{(49n^2-1)}{32n}\cdot\frac{x^{-4n}}{a^4c^4}+\text{etc.}$$

$$L=\frac{(3n-1)}{2}\frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ac}+\frac{(7n-1)}{2}\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3c^3}+\text{etc.}$$

$$M=1+\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\cdot\frac{x^{-2n}}{a^2c^2}+\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\frac{(49nn-1)}{32n}\cdot\frac{x^{-4n}}{a^4c^4}+\text{etc.}$$

$$N=\frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ac}+\frac{(nn-1)}{8n}\frac{(9nn-1)}{16n}\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3c^3}+\text{etc.}$$

Facto autem c negativo, erit alter valor particularis

$$y=-cx^{n-1}-\frac{(K-L)}{ax(M-N)}.$$

Erit ergo

$$P = \frac{acx^n(M+N)-K-L}{ax(M+N)}, \quad Q = \frac{-acx^n(M-N)-K+L}{ax(M-N)}$$

et $z:u = M+N : M-N$. Ex quibus colligitur, aequationis propositae

$$dy + ayydx = accx^{2n-2}dx$$

integrale completum fore:

$$Ce^{\frac{-2acx^n}{n}} = \frac{(acx^n - axy)(M+N)-K-L}{-(acx^n+axy)(M-N)-K+L}$$

sive $-C$ posito loco C

$$Ce^{\frac{-2acx^n}{n}} = \frac{ax(cx^{n-1}-y)(M+N)-K-L}{ax(cx^{n-1}+y)(M-N)+K-L}.$$

COROLLARIUM 2

8. Si cc est numerus negativus, fiet c hincque L et N quantitates imaginariae, at $c\sqrt{-1}$, $L\sqrt{-1}$ et $N\sqrt{-1}$ quantitates reales. Tum autem integrale completum realiter expressum erit:

$$C + \frac{acx^n}{n}\sqrt{-1} = A \text{ tang.} \frac{acx^nN - axyM - K}{acx^nM\sqrt{-1} - axyN\sqrt{-1} - L\sqrt{-1}}.$$

COROLLARIUM 3

9. Sit $c = b\sqrt{-1}$, ut habeatur haec aequatio integranda:

$$dy + ayydx + abbx^{2n-2}dx = 0.$$

Huius ergo aequationis integrale completum erit[1]):

$$C - \frac{abx^n}{n} = A \text{ tang.} \frac{abx^nN - axyM - K}{-abx^nM - axyN - L}$$

sive

$$C - \frac{abx^n}{n} = A \text{ tang.} \frac{K - abx^nN + axyM}{L + abx^nM + axyN};$$

H. D.

1) Vide p. 372 et 385.

existente

$$K=\frac{(n-1)}{2}-\frac{(5n-1)}{2}\cdot\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{x^{-2n}}{a^2b^2}+\frac{(9n-1)}{2}\cdot\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{(25nn-1)}{24n}\cdot\frac{(49nn-1)}{32n}\cdot\frac{x^{-4n}}{a^4b^4}-\text{etc.}$$

$$L=\frac{(3n-1)}{2}\cdot\frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ab}-\frac{(7n-1)}{2}\cdot\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3b^3}+\text{etc.}$$

$$M=1-\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{x^{-2n}}{a^2b^2}+\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{(25nn-1)}{24n}\cdot\frac{(49nn-1)}{32n}\cdot\frac{x^{-4n}}{a^4b^4}-\text{etc.}$$

$$N=\frac{(nn-1)}{8n}\cdot\frac{x^{-n}}{ab}-\frac{(nn-1)}{8n}\cdot\frac{(9nn-1)}{16n}\cdot\frac{(25nn-1)}{24n}\cdot\frac{x^{-3n}}{a^3b^3}+\text{etc.}$$

His igitur casibus integralia particularia, quae simul sint algebraica, non dantur.

COROLLARIUM 4

10. Quoties ergo fuerit $n=\frac{\mp 1}{2i+1}$, denotante i numerum quemcunque integrum, expressiones finitae algebraicae pro litteris K, L, M et N reperiuntur. His igitur casibus integratio aequationis huius

$$dy+ayydx=accx^{2n-2}dx$$

ope logarithmorum, huius vero aequationis

$$dy+ayydx+abbx^{2n-2}dx=0$$

ope quadraturae circuli absolvitur.

SCHOLION

11. Quoniam aequationis differentialis propositae $dy+ayydx=accx^{2n-2}dx$ integrale completum duplici modo expressimus, poterimus formulae integralis

$$\int\frac{e^{\frac{-2acx^n}{n}}dx}{zz},$$

quae in priori inest, valorem ex posteriori assignare, huiusque adeo integrationem, quae saepenumero maximopere difficilis videatur, exhibere. Posteriori modo autem invenimus

$$y = \frac{QR - P}{R - 1} = \frac{P - QR}{1 - R} = P + \frac{(P - Q)R}{1 - R},$$

at est

$$R = \frac{Ce^{\frac{-2acx^n}{n}} u}{z}, \quad P = cx^{n-1} + \frac{dz}{azdx} \quad \text{et} \quad Q = -cx^{n-1} + \frac{du}{audx}.$$

Consequenter habebitur

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{\left(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx}\right) Ce^{\frac{-2acx^n}{n}} u}{z - Ce^{\frac{-2acx^n}{n}} u}.$$

Per priorem vero integrationem est

$$y = cx^{n-1} + \frac{dz}{azdx} + \frac{e^{\frac{-2acx^n}{n}}}{zz \int e^{\frac{-2acx^n}{n}} adx : zz},$$

ex quorum comparatione oritur

$$\frac{z - Ce^{\frac{-2acx^n}{n}} u}{Czzu\left(2cx^{n-1} + \frac{dz}{azdx} - \frac{du}{audx}\right)} = \int \frac{e^{\frac{-2acx^n}{n}} adx}{zz},$$

quae transmutatur in hanc aequationem:

$$\frac{zdx - Ce^{\frac{-2acx^n}{n}} udx}{Cz(2acx^{n-1}uzdx + udz - zdu)} = \int \frac{e^{\frac{-2acx^n}{n}} dx}{zz}.$$

Quodsi ergo fuerit:

$$z = x^{\frac{-n+1}{2}} + \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)}{8n} \frac{(9nn-1)}{16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} + \text{etc.}$$

$$u = x^{\frac{-n+1}{2}} - \frac{(nn-1)}{8n} \cdot \frac{x^{\frac{-3n+1}{2}}}{ac} + \frac{(nn-1)}{8n} \frac{(9nn-1)}{16n} \cdot \frac{x^{\frac{-5n+1}{2}}}{a^2c^2} - \text{etc.}$$

haec formula differentialis

$$\frac{e^{\frac{-2acx^n}{n}}dx}{zz}$$

integrari poterit eritque integrale[1])

$$=\frac{zdx-Ce^{\frac{-2acx^n}{n}}udx}{Cz(2acx^{n-1}uzdx+udz-zdu)}.$$

Simili vero modo facto c negativo, quo z et u inter se permutantur, erit formulae differentialis

$$\frac{e^{\frac{+2acx^n}{n}}dx}{uu}$$

integrale

$$=\frac{udx-Ce^{\frac{+2acx^n}{n}}zdx}{Cu(-2acx^{n-1}uzdx+zdu-udz)}=\frac{Ce^{\frac{2acx^n}{n}}zdx-udx}{Cu(2acx^{n-1}uzdx+udz-zdu)},$$

in quibus integrationibus C denotat eam constantem arbitrariam, quae per integrationem more solito ingreditur.

1) Vide p. 404. H. D.

J. Liouville

REMARQUES NOUVELLES SUR L'ÉQUATION DE RICCATI

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COUNT RICCATI
AND
THE EARLY DAYS
OF THE
RICCATI
EQUATION
Sergio Bittanti ed.

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PURES ET APPLIQUÉES.

REMARQUES NOUVELLES

SUR

L'ÉQUATION DE RICCATI;

PAR J. LIOUVILLE.

[Présentées à l'Académie le 9 novembre 1840.]

L'équation différentielle du premier ordre, connue sous le nom d'équation de Riccati, a été l'objet des recherches d'un grand nombre de géomètres. Cette équation renferme dans son premier membre la somme de deux termes, l'un égal à la dérivée de la fonction principale y prise par rapport à la variable indépendante x, l'autre égal au produit du carré de y par une constante : dans le second membre, il entre un seul terme proportionnel à une puissance de la variable indépendante : l'exposant m de cette puissance peut être nommé *module* de l'équation.

Pour toutes les valeurs du module, on a trouvé la valeur complète de la fonction y exprimée sous forme finie à l'aide de quadratures définies. Mais lorsqu'on se borne à admettre dans le calcul les signes algébriques, exponentiels et logarithmiques, les cas d'intégrabilité deviennent très rares. Ceux que l'on a indiqués répondent à une certaine forme des valeurs du module m, savoir

$$m = -\frac{4i}{2i \pm 1};$$

on les a obtenus par des artifices particuliers, et les méthodes qui les ont fait connaître ne prouvent pas qu'ils soient les seuls possibles. On ignore complétement s'il y a d'autres valeurs du module m pour lesquelles la fonction y pourrait s'exprimer en x à l'aide d'un nombre limité de signes algébriques, exponentiels et logarithmiques. A la vérité les efforts réitérés des analystes pour découvrir quelque nouveau cas d'intégrabilité n'ayant produit aucun résultat, on est naturellement porté à croire qu'il n'en existe aucun différent de ceux que nous venons de citer. On conçoit pourtant que cela est loin de constituer une démonstration rigoureuse. J'ai donc pensé qu'il pouvait être bon de soumettre la question à une analyse exacte, et je suis parvenu à démontrer que les cas d'intégrabilité indiqués plus haut sont en effet les seuls admissibles. J'ajoute qu'il en serait encore ainsi lors même qu'aux signes algébriques, exponentiels et logarithmiques, on joindrait le signe $\int$ d'intégration indéfinie relative à la variable x.

1. On sait que l'équation de Riccati, savoir

$$(1) \qquad \frac{d\mathrm{y}}{d\mathrm{x}} + a\mathrm{y}^2 = b\mathrm{x}^m,$$

se ramène immédiatement à la forme

$$\frac{dv}{dz} + v^2 = z^m;$$

et qu'en posant

$$v = \frac{d.\log u}{dz} = \frac{1}{u}.\frac{du}{dz},$$

elle devient

$$(2) \qquad \frac{d^2u}{dz^2} = z^m.u.$$

Maintenant faisons

$$u = z^p.y,$$

ce qui nous donnera

$$z^p.\frac{d^2y}{dz^2} + 2pz^{p-1}.\frac{dy}{dz} + p(p-1)z^{p-2}.y = z^{m+p}.y,$$

puis substituons à la variable indépendante z une autre variable indépendante x, liée avec elle par la relation $z = x^r$. L'équation nouvelle entre y et x à laquelle nous serons conduits sera assez compliquée; mais on la simplifiera beaucoup en disposant convenablement des exposants indéterminés r et p. Nous prendrons

$$r = \frac{2}{m+2},\quad p = \frac{r-1}{2r} = -\frac{m}{4}.$$

De cette manière on aura

$$(3)\qquad \frac{d^2y}{dx^2} = \left(\frac{2}{m+2}\right)^2.\left[1 - \frac{m}{4}\left(\frac{m}{4}+1\right).\frac{1}{x^2}\right]y,$$

équation comprise dans la formule générale

$$(4)\qquad \frac{d^2y}{dx^2} = \left(A + \frac{B}{x^2}\right)y,$$

dont elle se déduit en posant

$$A = \left(\frac{2}{m+2}\right)^2,\quad B = -\left(\frac{2}{m+2}\right)^2.\frac{m}{4}\left(\frac{m}{4}+1\right).$$

A la vérité notre transformation devient impossible lorsque l'on a

$$m = -2,$$

mais alors l'équation (2) s'intègre immédiatement, et l'on trouve

$$u = Cz^q + C'z^{q'},$$

q et q' étant les deux racines de l'équation $q(q-1) = 1$, tandis que C et C' sont des constantes arbitraires. Nous mettrons de côté ce cas particulier : nous supposerons aussi que l'on n'a ni $m = 0$, ni $m = -4$, et que par suite B n'est pas zéro. Pour $m = 0$, l'équa-

1..

tion (2) fournirait

$$u = Ce^z + C'e^{-z};$$

pour $m = -4$, l'équation (3) se réduisant à

$$\frac{d^2y}{dx^2} = y,$$

s'intégrerait sans difficulté.

Si la valeur de Y peut s'exprimer en fonction de X à l'aide d'un nombre limité de signes algébriques, exponentiels et logarithmiques, et de signes d'intégrations indéfinies relatives à la variable indépendante X, il est visible que y devra s'exprimer de la même manière en fonction de x, et *vice versâ*. Au lieu de discuter en elle-même et directement l'équation de Riccati, il reviendra donc au même de discuter l'équation (4) à laquelle elle est intimement liée; c'est ce que nous allons faire. Pour plus de généralité, nous supposerons, dans nos premiers calculs, que les coefficients A et B ont des valeurs quelconques différentes de zéro, nous réservant de donner plus tard à ces coefficients les valeurs particulières qui conviennent à l'équation de Riccati.

2. Prouvons d'abord que l'équation (4) n'est vérifiée par aucune intégrale de la forme $y =$ *une fonction algébrique de* x, l'intégrale insignifiante $y = 0$ étant, bien entendu, mise de côté. Faisons

$$A + \frac{B}{x^4} = P,$$

en sorte que notre équation (4) devienne

$$\frac{d^2y}{dx^2} = Py. \tag{5}$$

Dès lors tous les raisonnements contenus dans les nos **4** et **5** de mon *Mémoire sur l'intégration d'une classe d'équations différentielles du second ordre en quantités finies explicites* [*], s'appliqueront à l'équation (5) sans qu'on ait besoin d'y changer un seul mot. Pour prouver

[*] Tome IV de ce Journal, page 423.

que l'équation (5) n'a pas d'intégrale algébrique, il suffit donc de prouver que le système d'équations suivant

$$(6)\quad \begin{cases} \dfrac{du}{dx} = u', \\ \dfrac{du'}{dx} = \mu P u + u'', \\ \dfrac{du''}{dx} = 2(\mu - 1) P u' + u''', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{du^{(i+1)}}{dx} = (i+1)(\mu - i) P u^{(i)} + u^{(i+2)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{du^{(\mu)}}{dx} = \mu P u^{(\mu-1)}, \end{cases}$$

ne peut jamais avoir lieu en prenant pour u une valeur rationnelle différente de zéro.

Si la valeur de u était rationnelle en x, celles de u', u'', etc., le seraient également. Imaginons que dans toutes ces fonctions et dans les produits Pu, Pu', etc., on ait séparé la partie entière de la partie fractionnaire, puis décomposé celle-ci en fractions simples par le procédé connu. Je dis qu'il n'entrera dans u aucune fraction dont le dénominateur soit de la forme $(x - x_1)^\gamma$, γ étant un nombre entier égal ou supérieur à l'unité, et x_1 une quantité différente de zéro; en effet, parmi toutes les fractions simples de la forme citée qu'on voudrait admettre dans la valeur de u, considérons spécialement celle pour laquelle l'exposant γ est le plus grand possible; à l'inspection des équations (6), nous voyons que le facteur $x - x_1$ devrait entrer dans les dénominateurs de u', u'', ..., $u^{(\mu-1)}$, $u^{(\mu)}$, avec les exposants respectifs $\gamma + 1$, $\gamma + 2$, ..., $\gamma + \mu - 1$, $\gamma + \mu$, ce qui est impossible puisque l'on a

$$\frac{du^{(\mu)}}{dx} = \mu P u^{(\mu-1)}.$$

Ce raisonnement cesserait d'être exact si l'on avait $x_1 = 0$, puisque P contient x en diviseur; nous voyons donc que le dénominateur de u

doit se réduire à une simple puissance de x, et que par suite il en est de même des dénominateurs de u', u'',... Toutes les quantités u, u', u'',..., peuvent donc s'exprimer par une suite de monomes formés chacun du produit d'une constante par une puissance de x égale à un nombre entier positif, nul ou négatif. Nous supposerons ces monomes ordonnés par rapport aux puissances descendantes de x, et pour mettre en évidence leurs premiers termes, nous écrirons

$$u = hx^n + \text{etc.}, \quad u' = h_1 x^{n_1} + \text{etc.}, \quad u'' = h_2 x^{n_2} + \text{etc.}$$

En même temps nous remplacerons la dernière des équations (6) par les deux équations équivalentes

$$\frac{du^{(\mu)}}{dx} = \mu \mathrm{P} u^{(\mu-1)} + u^{(\mu+1)}, \quad u^{(\mu+1)} = 0,$$

et nous chercherons les valeurs de $h_1, h_2, \ldots, h_{\mu+1}$: il faudra que l'on ait $h_{\mu+1} = 0$ pour que l'équation $u_{\mu+1} = 0$ soit satisfaite, et par conséquent pour que les équations (6) aient lieu avec une valeur rationnelle de u.

L'équation

$$u' = \frac{du}{dx}$$

donne d'abord

$$u' = nhx^{n-1} + \text{etc.};$$

d'où

$$n_1 = n - 1, \quad h_1 = nh.$$

L'équation

$$u'' = \frac{du'}{dx} - \mu \mathrm{P} u,$$

en observant que P est de la forme

$$\mathrm{P} = \mathrm{A} + \text{etc.},$$

donne ensuite

$$u'' = - \mu \mathrm{A} h x^n + \text{etc.};$$

d'où

$$n_2 = n, \quad h_2 = -\mu A h.$$

Ces premières valeurs sont comprises dans les formules générales

$$u^{(2q)} = (-1)^q . h . A^q . C_{2q} . x^n + \text{etc.},$$
$$u^{(2q+1)} = (-1)^q . h . A^q . C_{2q+1} . x^{n-1} + \text{etc.},$$

où C_{2q}, C_{2q+1}, sont des coefficients numériques essentiellement > 0. On s'assurera que ces formules sont exactes, en recourant à l'équation

$$u^{(i+2)} = \frac{du^{(i+1)}}{dx} - (i+1)(\mu - i) P u^{(i)}.$$

Pour $i = 2q$, cette équation donne

$$u^{(2q+2)} = (-1)^{q+1} . h . A^{q+1} C_{2q} . (2q+1)(\mu - 2q) . x^n + \text{etc.};$$

d'où résulte

$$C_{2q+2} = (2q+1)(\mu - 2q) C_{2q}.$$

Pour $i = 2q + 1$, elle fournit

$$u^{(2q+3)} = (-1)^{q+1} . h . A^{q+1} . (n C_{2q+2} + (2q+2)(\mu - 2q - 1) C_{2q+1}) . x^{n-1} + \text{etc.};$$

d'où

$$C_{2q+3} = n C_{2q+2} + (2q+2)(\mu - 2q - 1) C_{2q+1}.$$

Les valeurs obtenues ainsi pour C_{2q+2} et C_{2q+3} sont positives puisque l'indice $2q+2$ ou $2q+3$ ne surpasse jamais $(\mu + 1)$. La loi énoncée tout-à-l'heure est donc vérifiée : il s'ensuit évidemment que l'on ne peut avoir ni $h_{\mu+1} = 0$ ni $u^{(\mu+1)} = 0$. Donc l'équation (5) n'a jamais d'intégrale algébrique.

3. Maintenant appliquons à l'équation (5) tous les raisonnements contenus dans les n^{os} 9, 10 et 11 du Mémoire sur l'intégration des équations, déjà cité, et nous verrons que si l'équation (5) possède une intégrale de la forme $y =$ *une fonction finie explicite de x*, il faudra

nécessairement que l'équation

$$(7)\qquad \frac{dt}{dx} + t^2 = \mathrm{P}$$

puisse être satisfaite par une valeur de la forme $t =$ *fonction algébrique de* x. D'après la remarque faite au dernier numéro du même Mémoire, ce théorème subsisterait, et une valeur de t algébrique en x devrait encore satisfaire à l'équation (7), lors même qu'aux signes algébriques, exponentiels et logarithmiques, qui déjà sont supposés pouvoir entrer dans les fonctions finies explicites, on joindrait le signe $\int$ d'intégration indéfinie relative à la variable x. La recherche des cas où l'équation de Riccati peut s'intégrer sous forme finie en exprimant explicitement la fonction qui s'y trouve à l'aide de signes algébriques, exponentiels et logarithmiques, et de signes $\int$ d'intégrations relatives à la variable indépendante, se réduit donc à la recherche des cas où l'on peut trouverpour l'équation (7) une intégrale algébrique.

Mais on peut aller plus loin, car à l'aide des raisonnements contenus dans le n° **12** et dans la Note jointe au n° **15** du Mémoire sur l'intégration des équations [*], on prouve sans peine qu'une intégrale t de l'équation (7) ne peut pas être algébrique sans être en même temps rationnelle. C'est donc la détermination des intégrales rationnelles que l'équation (7) peut avoir qui doit à présent nous occuper.

4. La valeur de t que l'on cherche étant rationnelle, se composera naturellement d'une partie entière Q et d'une partie fractionnaire réductible en une série de fractions simples de la forme

$$\frac{\mathrm{G}}{(x-p)^{\alpha}},$$

de sorte que l'on peut écrire

$$t = \mathrm{Q} + \sum \frac{\mathrm{G}}{(x-p)^{\alpha}}.$$

De là résulte

$$\frac{dt}{dx} = \frac{d\mathrm{Q}}{dx} - \sum \frac{\alpha \mathrm{G}}{(x-p)^{\alpha+1}},$$

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[*] Tome IV de ce Journal, page 446.

et il faut qu'en portant ces valeurs dans l'équation

$$\frac{dt}{dx} + t^2 - P = 0,$$

celle-ci soit satisfaite. Mais comme son premier membre se composera, après la substitution, d'une partie entière et d'une partie fractionnaire, il sera nécessaire que ces deux parties s'annullent séparément. Or la partie entière de t^2 contient, outre Q^2, un terme Q_1 provenant du double produit de Q par la partie fractionnaire de t : la partie entière de P est d'ailleurs A. On doit donc poser

$$\frac{dQ}{dx} + Q^2 + Q_1 - A = 0.$$

Le degré de la fonction Q_1, et par suite de $\frac{dQ}{dx} + Q_1$, est inférieur au moins d'une unité à celui de Q. L'équation précédente exige donc que Q^2 et A soient de même degré : ainsi Q ne peut être qu'une simple constante; dès lors $\frac{dQ}{dx}$ et Q_1 doivent se réduire à zéro. Par suite on a nécessairement

$$Q^2 = A, \quad Q = \pm\sqrt{A}.$$

Maintenant occupons-nous de la partie fractionnaire de t. Parmi les fractions simples qui la composent, considérons spécialement celle qui répond au diviseur $x - p$, p étant différent de zéro, et, s'il y en a plusieurs de la forme

$$\frac{G}{(x-p)^\alpha},$$

attachons-nous de préférence à celle où l'exposant α est le plus grand. Dès lors les valeurs de $\frac{dt}{dx}$ et de t^2 pourront s'écrire ainsi

$$\frac{dt}{dx} = -\frac{\alpha G}{(x-p)^{\alpha+1}} + \frac{M_1}{N_1(x-p)^\alpha},$$

$$t^2 = \frac{G^2}{(x-p)^{2\alpha}} + \frac{M_2}{N_2(x-p)^{2\alpha-1}},$$

M_1, M_2, N_1, N_2, étant des polynomes dont les deux derniers ne sont pas divisibles par $x - p$. En les substituant dans l'équation (7), on aura,

$$\frac{G^2}{(x-p)^{2\alpha}} - \frac{\alpha G}{(x-p)^{\alpha+1}} + \frac{M_2}{N_2(x-p)^{2\alpha-1}} + \frac{M_1}{N_1(x-p)^{\alpha}} - P = 0.$$

Or 2α est au moins égal à $\alpha + 1$, et il y aurait absurdité à le supposer plus grand puisque alors la première fraction ne pourrait plus se réduire avec aucune autre; il faut donc poser $2\alpha = \alpha + 1$, c'est-à-dire $\alpha = 1$. Pour que les deux premiers termes se détruisent, G n'étant pas zéro, il faut ensuite que $G = 1$. Ainsi, abstraction faite des fractions simples dont le diviseur est monome, la quantité

$$\sum \frac{G}{(x-p)^{\alpha}}$$

est de la forme

$$\frac{1}{x-p} + \frac{1}{x-q} + \ldots\ldots + \frac{1}{x-r},$$

le nombre des fractions que nous venons d'écrire pouvant, bien entendu, se réduire à zéro.

Quant aux fractions dont le diviseur est un monome, on les mettra en évidence si l'on pose dans les calculs précédents $p = 0$, et si de plus on remplace P par sa valeur

$$A + \frac{B}{x^2}.$$

La dernière des équations obtenues plus haut devient alors

$$\frac{G^2}{x^{2\alpha}} - \frac{\alpha G}{x^{\alpha+1}} + \frac{M_2}{N_2 x^{2\alpha-1}} + \frac{M_1}{N_1 x^{\alpha}} - A - \frac{B}{x^2} = 0.$$

En prenant $\alpha > 1$, d'où $2\alpha > \alpha + 1 > 2$, on arriverait à une absurdité, puisque la première fraction ne se réduirait avec aucune autre. On doit donc poser $\alpha = 1$, et ensuite $G^2 - G = B$, pour que les termes qui ont le diviseur x^2 se détruisent. Il entre donc dans t une seule fraction de la forme $\frac{G}{x^{\alpha}}$; l'exposant α s'y trouve égal à 1, et le numérateur G est une des racines de l'équation $G^2 - G = B$; nous désignerons cette racine par $-\beta$, ce qui suppose

$$\beta(\beta + 1) = B.$$

On voit donc finalement que la valeur rationnelle de t, qui peut satisfaire à l'équation (7), doit être de la forme

$$t = \pm\sqrt{A} - \frac{\beta}{x} + \frac{1}{x-p} + \frac{1}{x-q} + \ldots + \frac{1}{x-r},$$

β satisfaisant, comme on l'a dit, à l'équation

$$\beta(\beta + 1) = B.$$

Puisque l'équation

$$(4) \qquad \frac{d^2y}{dx^2} = \left(A + \frac{B}{x^2}\right)y$$

est satisfaite par

$$y = e^{\int t dx},$$

nous devons en conclure que l'existence d'une telle valeur de t entraîne celle d'une valeur de y de la forme

$$y = e^{\pm x\sqrt{A}} . x^{-\beta} . Y,$$

Y étant un polynome entier

$$(x-p)(x-q)\ldots\ldots(x-r).$$

Ce polynome doit satisfaire à l'équation

$$(8) \qquad x\frac{d^2Y}{dx^2} - 2(\beta \mp x\sqrt{A})\frac{dY}{dx} \mp 2\beta\sqrt{A}.Y = 0,$$

que l'on déduit de l'équation (4) en y mettant pour y sa valeur et en remplaçant B par $\beta(\beta + 1)$.

2..

5. Désignons par n le degré du polynome Y, et posons

$$Y = x^n + h_1 x^{n-1} + \ldots + h_i x^{n-i} + \ldots + h_n.$$

En substituant cette valeur dans l'équation (8), puis égalant à zéro les coefficients des diverses puissances de x, on trouve d'abord $n = \beta$: cette équation, qui provient de l'égalité à zéro du coefficient de x^n, ne peut avoir lieu que si β est un nombre entier nul ou positif. La condition dont il s'agit étant supposée remplie et l'exposant n ayant été pris égal à β, on trouve ensuite généralement

$$h_{i+1} = \mp \frac{(\beta - i)(\beta + i + 1) h_i}{2\sqrt{A}.(i+1)},$$

formule à l'aide de laquelle on déterminera les coefficients successifs $h_1, h_2, \ldots, h_n$, en partant de $h_0 = 1$, et qui fournit, comme cela doit être, une valeur nulle de $h_{\beta+1}$ ou h_{n+1}. La condition que β soit un entier nul ou positif est donc la seule nécessaire; dès qu'elle est remplie, l'équation (8) est satisfaite par le polynome

$$Y = x^n + h_1 x^{n-1} + \ldots + h_n,$$

où $n = \beta$ et dont les coefficients sont déterminés par la formule que je viens d'indiquer.

Cela fournit en général pour Y une valeur double à cause du signe $\pm$ qui affecte le radical $\sqrt{A}$. En continuant à désigner par Y le polynome qui répond au signe $+$, et désignant par Z celui qui répond au signe $-$, on aura donc cette valeur complète de y,

$$y = Ce^{x\sqrt{A}}.x^{-\beta}.Y + C'e^{-x\sqrt{A}}.x^{-\beta}.Z,$$

où C et C′ sont des constantes arbitraires. Le cas où $\beta = 0$ fait exception, mais on peut ne pas s'en occuper puisqu'on a supposé d'avance que B n'est pas nul. Dans ce cas d'ailleurs l'équation (8) se réduisant à

$$x\frac{d^2Y}{dx^2} \pm 2x\sqrt{A}\frac{dY}{dx} = 0,$$

s'intègre immédiatement.

6. En résumé, pour que l'équation

$$(4) \qquad \frac{d^2y}{dx^2} = \left(A + \frac{B}{x^2}\right)y,$$

où la constante A est supposée essentiellement différente de zéro, puisse être satisfaite en prenant pour y une fonction de x exprimable par un nombre limité de signes algébriques, exponentiels et logarithmiques, et de signes $\int$ indiquant des intégrations indéfinies relatives à la variable x, il est nécessaire et suffisant que B soit de la forme $\beta(\beta+1)$, β étant un entier nul ou positif. Cette condition étant remplie, on pourra exprimer à l'aide des signes indiqués et même à l'aide des seuls signes algébriques et exponentiels l'intégrale complète de l'équation (4).

Voyons ce que devient la condition précédente pour l'équation de Riccati. On a alors

$$B = -\left(\frac{2}{m+2}\right)^2 \cdot \frac{m}{4}\left(\frac{m}{4}+1\right).$$

En remplaçant B par $\beta(\beta+1)$, ou $n(n+1)$, il vient

$$(2n+1)^2(m^2+4m) + 16n^2 + 16n = 0,$$

d'où l'on tire pour m ces deux valeurs

$$m = -\frac{4n}{2n+1}, \quad m = -\frac{4(n+1)}{2n+1},$$

dont la seconde peut être écrite ainsi

$$m = -\frac{4(n+1)}{2(n+1)-1},$$

et qui par conséquent peuvent être remplacées par la formule double si connue

$$m = -\frac{4i}{2i \pm 1},$$

où i désigne un nombre entier nul ou positif.

S. Bittanti

COUNT RICCATI AND THE EARLY DAYS OF THE RICCATI EQUATION

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COUNT RICCATI AND THE EARLY DAYS OF THE RICCATI EQUATION

Sergio Bittanti

1 Introduction

Towards the turn of the seventeenth century, when the baroque was giving way to the enlightenment, there lived in the Repubblic of Venice a gentleman, the father of nine children, by the name of Jacopo Franceso Riccati. In the cold New Year's Eve of 1720, he wrote a letter to his friend Giovanni Rizzetti, where, among other things, he proposed two new differential equations. In modern symbols, these equations can be written as follows:

$$\dot{x} = \alpha x^2 + \beta t^m \quad (1)$$

$$\dot{x} = \alpha x^2 + \beta t + \gamma t^2 \quad (2)$$

where m is a constant. This is probably the first document witnessing the early days of the Riccati Equation, an equation which was to become of paramount importance in the centuries to come.

2 A glimpse into Riccati's life

Count Jacopo Riccati was born in Venice on May 28, 1676. His father, a nobleman, died when he was only ten years old. The boy was raised by his mother, who did not marry again, and by a paternal uncle, who recognized unusual abilities in his nephew and persuaded Jacopo Francesco's mother to have him enter a Jesuit college in Brescia.

Young Riccati enrolled at this college in 1687, probably with no intention of ever becoming a scientist. Indeed, at the end of his studies at the college, in 1694 he enrolled at University of Padua as a student of law. However, following his natural inclination, he also attended classes of astronomy given by Father Stefano degli Angeli, a former pupil of Bonaventura Cavalieri. Father Stefano was fond of Isaac Newton's *Philosophiae Naturalis Principia*, which he passed on to young Riccati around 1695. This is probably the event which caused Riccati to turn from law to science.

After graduating on June 7, 1696, he married Elisabetta dei Conti d'Onigo on October 15, 1696. She bore him 18 children, of whom 9 survived childhood. Amongst them, Vincenzo (b.1707, d.1775), a mathematical physicist, and Giordano (b. 1709, d.1790), a scholar with many talents but with a special interest for architecture and music, are worth mentioning.

Riccati spent most of his life in his house of Castelfranco Veneto, a little town located in the beautiful country region surrounding Venice. Besides taking care of his family and his large estate, he was in charge of the administration of Castelfranco Veneto, as Provveditore of that town, for nine years during the period 1698 - 1729. He also owned a house in the nearby town of Treviso, where he moved after the death of his wife (1749), and where his children had been used to spend a good part of each year after 1747.

3 The "letterato" and the man

Notwithstanding all his responsibilities, Count Riccati always found time for his beloved studies. He did not follow any lecture courses in mathematics or other scientific disciplines. Basically, the profound knowledge of the self-taught man was acquired by reading. Among the journals of the period, it is worth mentioning the most important academic reports, in particular the *Commentari dell'Accademia delle Scienze di Bologna*, the *Acta Eruditorum Lipsiae* and the reports of St. Petersburg's Imperial Academy. Besides these scientific journals, Count Riccati was a reader of many Italian journals of general interest, e. g. the *Galleria di Minerva* and *Giornale de' Letterati d'Italia*, both printed in Venice, the former from 1696 to 1717 and the latter from 1710 to 1740. These journals had diverse interests; a typical issue would contain poems, short novels, philosophic essays and occasionally some mathematics. They reflected the spirit of the period prevailing in Northern Italy, according to which what we now consider a scientist was supposed to be a "*letterato*", a person of profound knowledge and vast interests. In line with this, Riccati had far-reaching interests, ranging from mathematics to poetry, from physics to religion, as witnessed by his works and his rich library. In conclusion, although he agreed with Galileo Galilei that "*la pietra lavagna e' la pietra di paragone degli ingegni*" ("the blackboard is the appropriate field of comparison of talents"), he also believed that the brain should be better exercised in a variety of fields. As he wrote: "*L'intelletto d'ogni uomo dovrebbe essere educato fin dalla sua adolescenza a far tesoro delle scienze piu' eccellenti e delle arti piu' belle. Non dico gia' che qualunque materia si debba scandagliare fino al fondo. Secondo il genio e il temperamento, una almeno se ne scelga e sopra di essa di proposito si metta studio. Nelle altre si faccia come la pecchia, che da ogni fiore va succhiando una qualche stilla, onde le cose e le voci per cui si esprimono non riescano nuove, ne' ci convenga restar su due piedi in molte occasioni, e guardare un silenzio poco onorevole, per non dir una decina di spropositi*" ("Since adolescence, the mind should be educated to treasure the most eminent of sciences and the finest of arts. I do not want to claim that every topic should be probed in detail. Following one's own talent and inclination, one should select at least one topic, and study it in depth. In the others, one should follow the example of the bee which sucks a drop of nectar from each flower...") (from *Opere*, Vol 1, p.164).

All through his life, an important complement of reading was direct exchange of ideas, through correspondence and conversation. Riccati was in contact with Domenica Maria Gaetana Agnesi, Gabriele Manfredi, Giovanni Poleni, Giovanni Rizzetti, Giuseppe Suzzi, Antonio Vallisneri, Bernardino Zendrini, and many other Italians. He was also in contact with various European mathematicians, such as Jacob Hermann and some members of the influential Bernoulli family, mainly Nicolaus III (b.1695, d.1726). Most of Riccati's correspondence can be found in Castelfranco Veneto, with the exception of the letters exchanged with members of the Bernoulli family, which are kept in the Basel University Library.

Riccati was an undemonstrative, kind man who preferred his home to academies and universities. His way of life was a very simple one, and he travelled very little. Probably, the only extended period he spent away from home was the summer of 1719, when, following the recommendation of his physician, he moved to Val di Sole to take advantage of the healthy water of that valley. He turned down many notable invitations, including the most appealing one of becoming president of St. Petersburg's Academy (c. 1725). He also refused the chair of Mathematics at the University of Padua and the invitation to the Court of Wien as Aulic Adviser. He was a member of the *Academy of Science of Bologna*, but he was informed of the appointment after his nomination.

Count Riccati was a strong and hard-working person, with an active and fertile mind throughout the years of his life. On April 2, 1754, he had a sudden bout of fever. A fortnight later, on April 15, he passed away.

4 Riccati and the academic world

While in various European Countries most scientists were already inclined to publish their results in short contributions as soon as they achieved them, the general attitude in Italy was to wait till a consistent amount of new results was achieved, and publish then a comprehensive book, an "*Opera*". There is no doubt that Count Riccati was influenced by this attitude, which perfectly matched his pacific nature. Another distinctive feature of Count Riccati's personality, so peculiar when compared with the general attitude of many academic scientists of his time, was his natural tendency to discuss freely the results of his achievements with friends and colleagues as soon as he obtained them, prior to any publication.

In this connection, the case of Maria Gaetana Agnesi (b.1718, d.1799), a mathematician of Milan, is worth elaborating. The oldest of 21 children, Maria Gaetana Agnesi was the daughter of a professor of mathematics, Pietro, who occupied a chair at the University of Bologna. Very early on, she was recognized as an infant prodigy, and she set out on her most important work, *Istituzioni Analitiche*, at the age of 20. She spent many years on this two volume treatise on differential and integral calculus. The *Istituzioni* were eventually published in 1748, with an immediate impact on the academic world. A masterpiece for its clarity and synthesis, Agnesi's book was translated into English by John Colson, professor at Cambridge, in 1801. In the preface to her *Istituzioni* she thanks Count Riccati for bringing to her knowledge an effective computational method of integrals which he had explained to her when the method was still unpublished. ("*Nel tomo secondo per entro il calcolo Integrale ritrovera' il lettore un Metodo affatto nuovo per i polinomi , ne' in luogo alcuno prodotto; questo e' del celebre, e mai abbastanza lodato Sig. Co. Jacopo Riccati Cavaliere di singolarissimo merito nelle scienze tutte, e ben noto al mondo letterario. Ha egli voluto fare a me questa grazia, che io non meritava, ed io rendo a lui, ed al pubblico quella giustizia, che si conviene*" - "In the second volume, when dealing with Integral Calculus, the reader will find a new method for polynomials; this is due to the famous Count Jacopo Riccati, a personality of unique merit in all sciences, and well known to the literate world. He was so kind as to favour me with such gift, which I did not deserve, and I now do justice to him, and to the public, as it should be.") Unfortunately, not all the scientists met by Riccati were as faithful in recognizing his merits as Maria Gaetana Agnesi was.

5 Riccati and differential equations

An appropriate way of appreciating Riccati's contribution to differential calculus is to consult his *Opere*, a work in 4 volumes published in Lucca by G. Rocchi in 1765, after Riccati's death. The editor was his son Giordano, to whom we are indebted for the care he took in collecting most of his father's works. However, if one is willing to follow the true sequence of Riccati's discoveries over the years of his life, one should complement the reading of his publications with the correspondence he wrote and received.

Riccati's main interest in the area of differential equations focused on the methods of separation of variables. Probably, such an interest originated in the reading of Gabriele Manfredi's book *De constructione aequationum differentialium primi gradus* printed in Bologna in 1707 (Manfredi occupied the Chair of Mathematics at Bologna University for many years). Riccati developed various methods, such as the method of *dimezzata separazione* (about 1715), and of *coefficienti ed esponenti indeterminati* (about 1717). Both methods were conceived to solve problems having a physical basis; in particular, the second one was motivated by a differential equation arising in a pendulum-type problem.

A compendium of Riccati's methods can be found in the lecture notes which he prepared for his private classes to Giuseppe Suzzi and Ludovico da Riva, who studied mathematics with him during 1722 and 1723. Subsequently, Suzzi and da Riva became professors of respectively physics and astronomy at the University of Padua. The lecture notes, which can be found in the *Opere*, are entitled *Della separazione delle indeterminate nelle equazioni differenziali di primo e di secondo grado, e della riduzione delle equazioni differenziali del secondo grado e d'altri gradi ulteriori* (On the separation of variables in differential equations of first and second order, and on the reduction of differential equations of second order and higher orders). The notes, actually 154 pages, are comprised of three parts and two appendices. The first and second parts are devoted to first order differential equations. Specifically, the first part (*Dei metodi inventati da vari celebri Autori per separare le indeterminate nelle equazioni differenziali di primo grado*) is devoted to methods for the separation of variables invented by other celebrated mathematicians. Here, special reference is made to the work of Gabriele Manfredi. In the second part (*Dei metodi inventati dall'autore per separare le indeterminate nelle equazioni differenziali di primo grado*), the methods of solution due to Riccati are discussed with reference to different equations which we would now call "Riccati equations." Finally, the third part deals with second order differential equations.

As for the specific equations he studied, in these lecture notes and elsewhere, we should mention, besides (1) and (2), a number of further equations of first order, such as

$$\dot{x} = \alpha\, t^p\, x^2 + \beta\, t^m \tag{3}$$

$$\dot{x} = \alpha\, t^p\, x^q + \beta\, t^m \tag{4}$$

where m, p, and q are constants. Obviously, (1) is a particular case of (3) (with $p = 0$) and (3) is a particular case of (4) (with $q = 2$). As for the second order equations, he was particularly attracted by the equation

$$\ddot{x} = \alpha\, t^m ,$$

which he called "*equazione ingannatrice*" ("misleading equation"). A similar appreciation for such an equation was also expressed by Euler some years later.

6 History and prehistory of the Riccati equation

Most probably, Riccati begun to conceive and study eqs. (1) - (4) in 1715. Unfortunately, in agreement with his general attitude, he did not immediately publish the fruit of his work. To the best of our knowledge, the first evidence of these studies goes back to 1719, when Riccati met Nicolaus II Bernoulli (1687-1759), who lived in Padua then (Nicolaus II held the Chair of Mathematics at the University of Padua from 1716 to 1719). In this meeting, he brought to the attention of this member of the Bernoulli family "his" differential equations, and the methods of solutions he knew. In particular, he raised the question of finding those triples (m,p,q) for which it was possible to separate the variables.

On April 1, 1719, Nicolaus II wrote an interesting letter to Pierre Remond de Monfort, where he discusses the cases of separation of variables of which he became aware during this private communication, in particular the case when $q = 2$ and $m + 3p = -4$. A second reference to such a conversation can be found in a letter , dated February 5, 1721, written by Riccati to another member of the Bernoulli family, Nicolaus III, cousin of Nicolaus II.

During the years 1720 and 1721, Nicolaus III was in Venice as a tutor in a noble family. He and Riccati met a few times, and then exchanged a conspicuous correspondence during that period , and following it. The contact between the two was probably produced by the letter we referred to at the beginning of this paper, the letter Riccati wrote on January 1, 1721, to Giovanni Rizzetti. In this important document, Riccati makes reference to a meeting he had in Bologna with Gabriele Manfredi, with whom he discussed various questions, in particular the problem of the separation of variables in differential equations. Then he writes, "*When I came back home, I began to write all that I had thought of on this subject, but I was delayed by two reasons, namely the difficulty in computation and the suspicion that what I believed to be new was already known to analysts of great experience, such as the members of Bernoulli family. This is why I would like to ask you to be so kind as to communicate the two subsequent formulas, which are among the simplest ones encompassed by my method, to the Eruditissimo Signor Niccolo' Bernullj...*" After writing eqs. (1) and (2), Riccati continues as follows, "*Untill now, I have not been able to separate the variables in the above formulas in general, and I don't know whether it will be possible. However, I have found infinite values of the integer m for which the variables can be separated...*".

From the subsequent letters written by Nicolaus III to Riccati, it is apparent that Nicolaus III encountered many difficulties in proving that the variables were separable for Riccati's sequence of integers m. Furthermore, it appears that, sometime in 1721, Riccati communicated verbally this famous sequence to Nicolaus III. Indeed, in Nicolaus III's letter to Riccati dated August 26, 1721, one can read, "*I have eventually found the same cases of separation of variables you referred to when we met in Val di Sole*". Then, he presents his line of reasoning to prove that in equation (3) the separation was possible for m = -3p -4 and m = (-p - 4) / 3. This sequence is also reported on in the second part of Riccati's lecture notes on the separation of variables mentioned above.

In the same letter, Nicolaus III invites Riccati to write a paper for the *Acta Eruditorum Lipsiae* to pose his problem to all mathematicians, "*who, after the death of Leibnitz and others, had begun to languish.*"

Riccati accepted the suggestion of Nicolaus III and, in the Supplements of that Journal, Vol. 8, Sect. 2, pages 67 -73, 1724, the paper *Animadversiones in Aequationes differentiales secundi gradus* was published. In this seven-page paper written in Latin, Riccati reduces a second order differential equation to equation (3). Then he poses the problem of finding all exponents m and p for which the separation of variables was possible. More precisely , the last statement of the paper is:

In superiori formula $x^m dq = du + uu\, dx:q$, *dato ad libitum exponente m, statuatur quantitas* $q = x^n$. *Peto qua ratione determinandi sint valores alterius exponentis n, ut succedat indeterminatarum separatio, & aequationis constructio per solas quadraturas.*

(In the previous formula $x^m dq = du + uu\, dx:q$, given any exponent m, let $q=x^n$. I pose the question of finding those values of the exponent n such that the separation of variables is possible...). Note that, in the terminology of the period, u^2 is indicated as uu and differentials are used in place of derivatives. Moreover, the symbol : indicates division.

This paper, published in 1724, is usually considered as the first official document on the Riccati equation. Strangely enough, however, in 1723, in the Supplements of the same Journal, pages 503 - 510, an *Appendix* to the above paper was published. In other words, the appendix was published prior to the paper! The reason for this remains a mystery. As an obvious consequence, however, we can conclude that Riccati's original paper, written on suggestion of the letter of August 1721 of Nicolaus III, was probably submitted in 1721 or 1722, although its publication was postponed until 1724.

7 The Riccati equation over the centuries

As we have already said, Riccati knew infinitely many cases when the separation of variables was possible in his equations. Apparently, he did not know whether his solution was the most general one. In the paper of 1724 in the *Acta Eruditorum Lipsiae*, interested as he was in posing the problem in its full generality, Riccati did not even mention the infinite sequence he communicated to Nicolaus III in 1721. In the same issue of the *Acta Eruditorum Lipsiae*, Riccati's paper was immediately followed by a comment by Daniel I Bernoulli (Daniel I (b.1700, d. 1782) was a brother of Nicolaus III). The comment ends as follows: "*I now add the solution* (to Riccati's problem). *However, in order to leave to others the possibility of attempting, I will supply it in a disguised form, the significance of which will be clarified in due time. The solution of the problem posed by Riccati Esq.in disguised form is:*

14a, 6b, 6c, 8d, 33e, 5f, 2g,4b,33i,6l, 21m, 26n, 16 o, 8p, 5q, 17r, 16s, 25t, 32u, 5x, 3y, +,-,———,±, = , 4, 2, ? "

It is not clear to me whether the last character (above indicated as ?) is the *number 1* or the *letter l*. We also don't know whether Bernoulli's riddle has ever been solved by anyone.

During the latter half of the second decade of the eighteenth century, various authors studied Riccati problem in its full generality, mainly his pupil Giuseppe Suzzi, Nicolaus III and, as already said, Daniel I Bernoulli. With reference to equation (1) Giuseppe Suzzi and Daniel I derived the formula $m = -4k / (2k \pm 1)$, supplying, for all integers k, infinitely many cases when the variables can be separated. Nicolaus III considered the more general equation (3) and derived the formula $m = (-2kp - 4k \pm p)/(2k \pm 1)$, where k is any positive integer. Here, letting $p = 0$, the sequence discovered by Suzzi and Daniel I is obtained. Moreover, by letting $k = 0$ and taking - in the ±, the Riccati sequence $m = -3p - 4$ follows. Some years later, in 1732, Euler obtained the same result by an ingenious procedure of integration by series applied to equation (1). He also applied the continued fraction method to the equation and its generalizations. As a matter of fact, the sequence $m = -4k / (2k \pm 1)$ provides all (and the only) integers for which the variables can be separated in equation (1), as proven in 1841 by Liouville. In 1776, Lagrange proposed the scalar and time-varying Riccati equation in its full generality.

Many scientists have studied the reduction of differential equations of higher order to a Riccati equation. In particular, in 1851, Francesco Brioschi, the first Rector of the Politecnico di Milano, reduced a very general second order differential equation to a Riccati equation. Analogous studies were performed by various physicists in connection with different problems. For instance, according to a quotation of Edoardo Amaldi, Ettore Majorana analyzed the Thomas-Fermi equation via the Riccati equation.

The importance of the Riccati equation in modern times need hardly be emphasized. The equation is now fundamental for many design problems in engineering, especially filtering and control.

As for the theoretical analysis of the equation, the emergence of system theory concepts in the second part of the twentieth century in the control science community has led to a breakthrough. Notions such as controllability and observability, stabilizability and detectability, have been found to be important tools to study the equation, and many properties previously obscure have been eventually brought to light. Attention has also been paid to the discrete-time counterpart of the Riccati equation, a difference equation usually also named after Riccati. Efficient numerical algorithms to solve these equations, both in continuous and discrete time, are now available.

This very active research of the last 30 years will undoubtedly continue to be so for many decades to come, and we can only wonder what new developments will be brought by the forthcoming century on this venerable equation, conceived by Count Riccati more than a quarter of a millennium ago.

Acknowledgement

My warm thanks to Professors A.J. Laub, A. Locatelli, M. Maranzana-Figini and J.C. Willems who were kind enough to read and comment an early draft. The views expressed, needless to say, are my own.

For their help in the collection and joint reading of historical papers, I am grateful to Doctor A. Bazzeo, Monsignor A. Campagner, Doctors M. Campi and G. De Nicolao and Professor E. Knobloch. I also had interesting discussions with Doctor L. Grugnetti, who was so kind as to pass me a copy of the letter by Count Riccati to Nicolaus III reported in Appendix 2, as well as further historical documents.

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Appendix 1

Cover page of a volume of the "Opere"

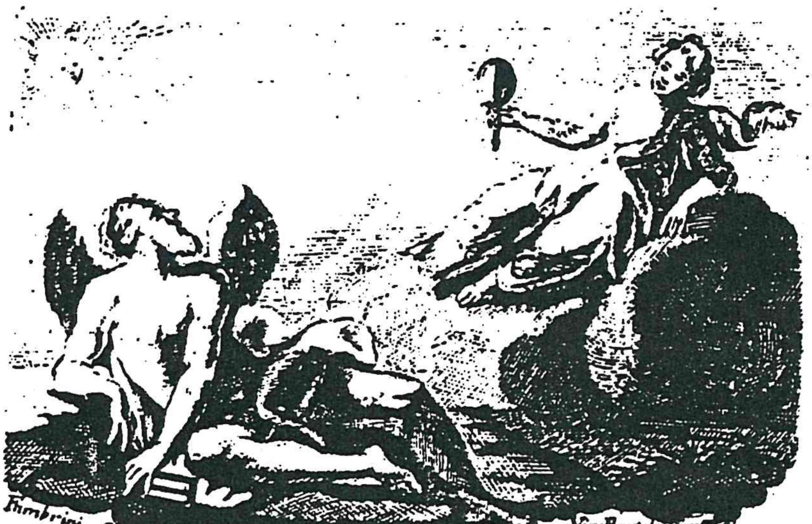
OPERE

DEL CONTE

JACOPO RICCATI

NOBILE TREVIGIANO.

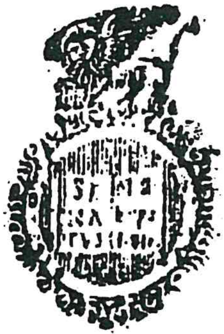
Tomo Quarto.



IN LUCCA MDCCLXV.

APPRESSO GIUSEPPE ROCCHI

CON LICENZA DE'SUPERIORI.



Appendix 2

Letter written by Riccati to Nicolaus III Bernoulli on December 2, 1721.

Ill.mo Sig. Sig. Col.mo 189

Castelfranco li 2 Xbre 1721

Se varie occupazioni non mi avessero distratto, spedirei a V.S. Ill.ma con la presente occasione la lettera diretta al Sig. suo Padre; ma non avendo avuto tempo di finirla, è d'uopo, ch'io la rimetta al primo incontro.

Io non ho più avuto risposta alla nostra lettera del Sig. Dr. Michelotti. La disputa versa sopra il secondo Corollario della Prop. 36 del Lib. 2° de' Principj del Sig. Newton. Io lo giudico vero, ed il Sig. Michelotti falso. Qui non v'è mezzo, uno di noi due certam.te s'inganna. Non so, se a Basilea abbiano notizia di questa picciola controversia, nè di qual opinione siano: comunque sia [illegible] il vero, penso, che mi siano Avversarj; stantechè la risposta [illegible] dal Sig. Michelotti viene certam.te da altra penna, e non dalla sua. Quand'avrò stesa la mia replica la farò vedere a lei, per altro avrei soddisfazione d'intender [illegible] del loro, perchè gl'ingegni non si rendessero più baldanzosi. L'opposizione [illegible] certam.te ingiu-

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* (A copy of this letter was passed to the author from Dr. L. Grugnetti).

circo anche a loro, né mancheranno di farlo ancora. . . .

Non posso dire a SS. Ill.ma il nome di chi mi ha mandato la formola conseguenza; mentre più ora almeno vuole esser incognito.

Non senza mistero è stata proposta, e se qualche cosa io dicessi di più alla saviezza mi capirebbe.

In questi ultimi giorni l'ho considerata, e trovo come prima giudico, essere questa una equazione ingannatrice, che non può integrarsi senza supporre qualche quantità costante, la quale variata, si diversifica la natura della curva: cosicchè ogni curva soddisfarebbe; quando si possa prendere la costante d'arbitrio, ed all'opposto nessuna, se niuna quantità come costante si prende. Io non nego, che alla ~~[illegible]~~ credenza questo meriti una decisione. Propongo la mia opinione, e ne lascio a Lei il giudizio.

Per darne un saggio semplicissimo, prendo in mano un caso particolare della formula, supponendo $xddx = ddy$. Faccio $x = z^m$, onde sia $dx = mz^{m-1}dz$, e posta dz come costante

$ddx = \overline{mm-m}, z^{m-2} dz^2$. Fatta la detta sostituzione, averemo $\overline{mm-m}, z^{2m-2} dz^2 = ddy$. Pongo $dy = pdz + Bdz$, e prese le differenze seconde trattando come si deve costante dz, nascerà un'altra equazione $\overline{mm-m}, z^{2m-2} dz^2 = dpdz$. ovvero $\overline{mm-m}, z^{2m-2} dz = dp$. e integrando $\frac{mm-m}{2m-1} z^{2m-1} = p$. ommessa la costante che potrebbe aggiungersi per ultimare il calcolo, e trascurato anche il coefficiente, sarà $z = p^{\frac{1}{2m-1}}$, e $z^m = p^{\frac{m}{2m-1}}$ ma $dy = pdz + Bdz$; dunque $dy = z^{2m-1} dz + Bdz$. e di nuovo integrando ommessi i coefficienti $y = z^{2m} + bz$. e posto in vece di z il suo valore dato per x. risulterà $y = x^2 + bx^{\frac{1}{m}}$ la qual equazione abbraccia curve infinite, secondo il vario valore di m.

Sia di nuovo l'equazione $xddx = ddy$. Pongo $x = z + z^2$. e differenziando $dx = dz + 2zdz$. Prendo le seconde differenze posta dz costante; dunque $ddx = 2dz^2$, e in conseguenza $xddx = 2zdz^2 + 2z^2dz^2 = ddy$. Sia $dy = pdz + Bdz$. dunque $ddy = dpdz$; onde $2zdz^2 + 2z^2dz^2 = dpdz$, e $2zdz + 2z^2dz = dp$.

sostituendo. Dunque avrassi $z^2 + \frac{2z^3}{3} = p$. ma $dy = p\,dz + b\,dz$

Quindi $dy = z^2dz + \frac{2z^3dz}{3} + b\,dz$, e integrando

$$y = \frac{z^3}{3} + \frac{z^4}{6} + bz.$$

ed eccone un'altra curva totalm.te diversa, e con lo stesso metodo altre infinite potrebbono scoprirsi, le quali tutte muterebbero la costante attuale trasformando. Io non vo poi cercar quelle, che soddisfaciano, senza prendere alcuna quantità come costante.

Mi farà favore V.S. Ill.ma di dirmi sopra ciò il suo parere; mentre V.S. rassegnandole il mio rispetto, mi dico

Di V.S. Ill.ma

Affezionatiss.o Ser.e
Giacomo Riccati

References

A very detailed report of 67 pages on Riccati's life and work is given in

C. di Rovero . *Vita del Conte Jacopo Riccati.* in *Opere del Conte Jacopo Riccati nobile trevigiano.* G. Rocchi. Lucca 1765.

This paper was written soon after Riccati's death, by an author who had the privilege of knowing Riccati personally.

A very good report is

A. A. Michieli. *Una famiglia di matematici e di poligrafi trivigiani: i Riccati.* in *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti.* Tomo 102, Parte ii, 1942-43.

Further useful references of general interest are (in chronological order):

G.B. Marzari. *Elogio di Jacopo Riccati pronunciato nella grand'aula del regio liceo del Tagliamento per il riaprimento degli studi il 15 Novembre 1812.* G. Trento e figli, Treviso, 1813.

Cantor M. *Vorlesungen hüber Geschichte der Mathematik,* Band iii, Leipzig Teubner, 1901.

I. Szabo. *Die Familie der Mathematiker Riccati.* I Mitteilung. in *Humanismus und Technik,* Berlin 1974, p.37-75; II Mitteilung in *Humanismus und Technik,* Berlin 1974, p. 109 - 131.

L. Grugnetti. *Sulla vecchia e attuale equazione di Riccati.* in *Rendiconti del Seminario Facolta' di Scienze dell'Universita' di Cagliari,* Vol. 55, Fasc. 1, 1985, p. 7 - 23.

One of most interesting papers by Euler on the Riccati equation is the following one, written in Latin.

L. Euler. *De risolutione aequationis* $dy + a\, yydx = bx^m dx$. Novi Commentari academiae scientiarum Petropolitanae 9 (1762/3, 1764), p. 154 - 169.

Euler's work on the Riccati equation and other equations can be appreciated by the recent translation from Latin due to M.F. Wyman and B.F. Wyman:

L. Euler. *An Essay on Continued Fractions*. in *Mathematical Systems Theory*, Vol. 18, p. 295-328, 1985 (translation of the original paper by Euler published in 1744).

Liouville's proof that $m=-4k/(2k \pm 1)$ is the only sequence for which the integration of equation (1) is possible can be found in:

J. Liouville. *L'equation de Riccati*. in *Journal de Mathématiques pures et appliqées*. Tome VI, p. 1-13, 1841.

The work by Francesco Brioschi, first Rector of the Politecnico di Milano, on the reduction of second order differential equations to the Riccati equation is given by:

F.Brioschi. *Intorno la integrazione di una equazione alle derivate del second'ordine*. Annali Scienze Matematiche e Fisiche, Tomo II, p. 497-502, 1851.

Majorana's interest in Riccati equations is documented by Edoardo Amaldi in

L.Sciascia. *La scomparsa di Majorana*. Einaudi, 1975.

The famous letter written by Riccati to Giovanni Rizzetti dated January 1st, 1721, as well as five letters to Nicolaus III Bernoulli, can be found in

L. Grugnetti. *L'Equazione di Riccati*. Bollettino di Storia delle Matematiche, Vol VI, fasc. 1, p. 45- 82, 1986.

The above letters witness the very early days of the Riccati equation. In Grugnetti's paper, one can also find 6 letters written by Nicolaus III to Riccati in 1721, during the period of his stay in Italy, plus two letters written by Nicolaus III from St. Petersburg in 1726, one to Riccati and one to Gabriele Manfredi. Most correspondence is in Italian, except one letter in Latin.

The remarkable life of Domenica Maria Gaetana Agnesi is reported in many books, such as

L.M. Osen. *Women in Mathematics*. The MIT Press, Cambridge, 1974.

As general reference for the history of mathematics, the author used

C.B. Boyer. *The History of Mathematics*. J. Wiley and Sons, 1974.

In particular, the terminology for the members of the Bernoulli family conforms to that of Boyer.

More than a quarter of millennium ago, Jacopo Francesco Riccati, a nobleman born in Venice in 1676, conceived and studied a first order differential equation which is now universally named after him.

The original Riccati's paper published in 1724 in the *Acta Eroditorum Lipsiae* is reproduced herein. In this paper, Riccati reduces a second order differential equation to a first order one, an equation which we would now call a Riccati Equation. Since the known term of this equation is a power of the independent variable, the question of finding all exponents for which the separation of variables is possible was posed at the end of the paper. Actually, from various documents it is apparent that Riccati already knew infinitely many cases for which the separation of variables was possible.

Later, Riccati's problem was considered by many mathematicians, such as Euler and Liouville. Here, two most interesting papers of these scientists are reproduced.

The booklet ends with a paper on Count Riccati and the early days of his equation, where the history and prehistory of the Riccati equation can be found.